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Unità cognitiva

In Italy (as in other countries) the usual school approach to theorems concerns plane Euclidean geometry theorems; students are provided with some examples of statements and their proofs to understand and repeat; at a further stage, some statements are presented to students, with the task of proving them. Most teachers do not engage students in producing conjectures. This school approach to theorems is very hard for students; in the past it was one of the main reasons for drop out at the entrance to scientific oriented high schools in Italy. As a consequence, in the last decades many teachers progressively reduced the importance of (or postponed) the activities concerning theorems in high school.

When considering this kind of difficulties a preliminary question must be posed: does the failure depend on the lack of some specific, high level intellectual qualities, or does it depend on the didactical choices? In the case of the approach to theorems, preceding studies had shown that the usual approach to proof in the case of plane Euclidean geometry theorems is difficult to motivate for students: most statements can be easily checked by measuring, and several of them are also "evident".

Moreover, when students do engage in proving an intriguing statement presented to them, they tell that they have an "empty mind". Teachers (and student themselves) think that the only possibility is to learn the proofs presented by the teacher. Repeating proofs becomes a kind of magic practice performed to satisfy the contract with the teacher.

From a cultural point of view, several students think that proof is depending on teacher's authority, not the expression of a rationality depending on rules determined by the historical evolution of mathematics but independent from authoritarian relationships: "Proof is valid if the teacher says that it is so" (16% of answers from a questionnaire at the entrance to the University!)

In order to modify this situation, radical changes in the educational and research perspectives are needed. Conjecturing must become the source of the need for proving (and this poses the research problem of the choice of contexts where conjecturing can be authentic, with a real need for intellectual arguments in order to escape uncertainty situations). In order to choose suitable conjecturing tasks, knowledge about the conjecturing process must be increased. In order to avoid the "empty mind" feeling during the approach to proof, research should elaborate on the working hypothesis that beginners' proving must be rooted in the argumentative activity consisting in the search and elaboration of arguments for the plausibility of the conjecture.

1. The description of some characteristics of contexts and tasks that are suitable for beginners' conjecturing and proving.

Concerning conjecturing, the challenge is to create a situation of uncertainty, where some regularities can be hypothesised (or appropriated as reasonable) by students through suitable explorations, favoured by the familiarity with the context, especially as concerns mental behaviours inherent in ordinary experience of it. Concerning subsequent proving, the challenge is to prevent students from unacceptable arguments (i.e. those deriving from measuring, or considering special cases that provide sufficient evidence for the validity of the conjecture, etc.).

An example of satisfactory context and task is the following: in the experience field of "Representation of visible space", VII-grade students are considering how rectangular, circular, etc. thin metal sheet can be seen (and drawn "as we see them"). An appropriate question can be: "What does it happen with a rectangular sheet? Can we see (and draw "as we see") it as a non rectangular parallelogram?".

2. The characterisation of some components of the conjecturing process.

Different kinds of inference, intervening (alone or combined) in the conjecturing process:

- inference based on induction;

- inference based on abduction;

2. The characterisation of some components of the conjecturing process.

- inference based on a temporal section of a dynamic exploration process (during the exploration, I perceive a regularity in coincidence with a condition that is satisfied in that very moment);

- inference based on a temporal expansion of a regularity (I move from a known regularity, to an exploration aimed at discovering what conditions allow to maintain the regularity).

I. 3. The characterisation of "cognitive unity of theorems" as that peculiar situation where some arguments, produced for the plausibility of the conjecture during the conjecture production (or appropriation) phase, become ingredients for the construction of proof.

Cognitive unity of theorems concerns the possible continuity between some aspects of the conjecturing process and some aspects of the proving process.

First of all, the arguments.
During the conjecturing phase some relations, known properties, evidences, general rules, etc. can be produced or evoked as reasons for the plausibility of the conjecture. Some of these arguments can intervene in the proving process as relevant arguments to support further findings (e.g. generalisations), or as components of the final deductive reasoning.

But cognitive unity concerns also those conditions that allow some arguments, produced during the conjecturing phase, to be exploited during the proving phase:

in particular, the continuity of the mathematical frame (most arguments produced during conjecturing within a synthetic geometry frame are difficult to recycle in the proving phase within an analytic geometry frame);

the continuity of the exploration strategies and heuristics (arguments which are relevant in a given exploration during the conjecturing phase may become useless, or even be forgotten, in another kind of exploration during the proving phase);

the continuity of the external representation - an important change in the external representation between the conjecturing phase and the proving phase can make unavailable all the arguments that are strictly related to a peculiar representation - for instance, visual arguments related to graphs of functions can become unavailable when we use algebraic language to deal with the same problem situation.

As such, "cognitive unity of theorems" focus on the arguments and related conditions of stability in the transition from conjecturing to proving, not on the possible "structural" analogy, or "continuity", between the argumentation during the different phases of the activity which brings from the search of a conjecture to the text of a proof.

Very frequently it happens that the "reference system continuity" is kept, while the "structural continuity" is broken.

I. Some experimental evidence for the hypothesis that in a situation of potential "cognitive unity of theorems" (intended as "reference system continuity") most VIII-graders (and, more generaly, most beginners not yet conditioned by negative impacts with proving) are able to produce proofs.

I. Some possible ways to ensure the transition from the “naive”, yet cognitively consistent experience of conjecturing and proving to the “culture of theorems”, where statements and proofs take a precise historically situated and socially shared meaning, and typical reasoning forms (e.g. “reductio ad absurdum”, or specific forms inherent in specific fields of mathematics) are available.

We can consider two different perspectives:

- that of “reference system continuity”
- that of “structural continuity”