SIS Piemonte
a.a. 2004_2005

Corso di Fondamenti della Matematica

Nodi fondamentali in Matematica

2° incontro

Le funzioni

Overarching ideas

- Cambiamento e relazioni
- Spazio e forma
- Quantità
- Incertezza

Change and relationships involves mathematical manifestations of change as well as functional relationships and dependency among variables. This content area relates most closely to algebra. Mathematical relationships are often expressed as equations or inequalities, but relationships of a more general nature (*e.g.,* equivalence, divisibility and inclusion, to mention but a few) are relevant as well. Relationships are given a variety of different representations, including symbolic, algebraic, graphic, tabular and geometric representations. Since different representationsmay serve different purposes and have different properties, translation betweenrepresentations is often of key importance in dealing with situations and tasks.

(PISA)

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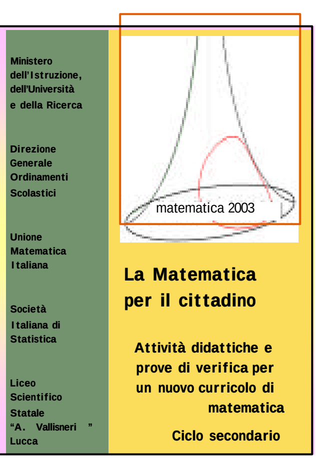
nuclei tematici:

- *Numeri e Algoritmi*
- *Spazio e figure*
- *Relazioni e funzioni*
- *Dati e previsioni*



Alcuni strumenti epistemologici e cognitivi per un approccio didattico al concetto di funzione

FUNZIONI → ALGEBRA

FUNZIONI → CALCOLO

ESEMPIO:

La formula del problema sulla moneta e la scacchiera:

$$p = \frac{(L-2R)^2}{L^2}$$

$$p = \frac{(L-2R)^2}{L^2} = \left(\frac{L-d}{L}\right)^2 = \left(\frac{h}{L}\right)^2 = k^2$$

Rather than deal initially with formal definitions which contain elements unfamiliar to the learner, it is preferable to attempt to find an approach which builds on concepts which have the dual role of being familiar to the students and also provide the basis for later mathematical development.

Such a concept I term a *cognitive root*. These are not easy to find - they require a combination of empirical research (to find out what is appropriate to the student at the current stage of development) and mathematical knowledge (to be certain of the long-term mathematical relevance). A cognitive root is different from a mathematical foundation. Whilst a mathematical foundation is an appropriate starting point for a logical development of the subject, a cognitive root is more appropriate for curriculum development.

(D. Tall)

COGNITIVE ROOT

A cognitive root is a concept that:

(i) is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence,

(ii) allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,

(iii) contains the possibility of long-term meaning in later developments,

(iv) is robust enough to remain useful as more sophisticated understanding develops.

(Tall, 1989; Tall, McGowen and DeMarois, 2000)

The function concept, according to Kleiner (1989), "goes back 4000 years; 3700 of these consist of anticipations". Its evolution has led to a complex network of conceptions:

- **the geometric image of a graph,**
- **the algebraic expression as a formula,**
- **the relationship between dependent and independent variables,**
- **an input-output machine allowing more general relationships,**
- **the modern set-theoretic definition.**

N. Bourbaki, *Éléments de mathématique*, Hermann, 1970

Déf. 9. ... une correspondance f = (F,A,B) est un fonction si, pour tout x appartenant à l'ensemble de départ A de f, la relation (x,y) e F est fonctionelle en y; l'objet unique correspondant à x par f s'appelle la valeur de f pour l'élément x de A, et se désigne par f(x)... (II, Théorie des ensembles, §3, n°4)

Una corrispondenza f=(f,a,b) e' una funzione se, per ogni x appartenente all'insieme di partenza A di f, la relazione (x,y) e F e' funzionale in y; l'unico oggetto corrispondente a x tramite f si chiama valore di f per l'elemento x di A e si indica con f(x).

However, there is much empirical evidence to show that, though this definition is an excellent mathematical foundation, it may not be a good cognitive root.

Malik (1980) highlighted the manner in which this definition represents a very different frame of thought from that experienced in traditional calculus emphasising the rule based relationship between a dependent and independent variable.

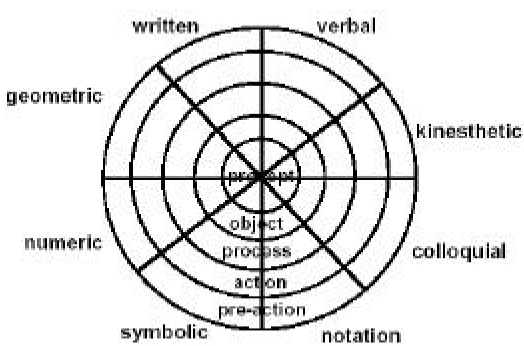
"The most fundamental conception of a function is that of a relationship between variable magnitudes. If this is not developed, representations such as equations and graphs lose their meaning and become isolated from one another...Introducing functions to young students by their elaborate modern definition is a didactical error - an antididactical inversion." (Sierpinska, 1988, p. 572)

Empirical research shows that, even when students are given such a formal definition, their overwhelming experience from examples of functions with implicit common properties causes them to develop a personal concept image of a function which implicitly has these properties. For instance, if the functions encountered are given mainly in terms of formulae, this causes many students to believe that the existence of a formula is essential for a function.

The research shows a wide variety of approaches to the complexity of the function concept.

Some gain can be made in improving understanding and problem -solving abilities in specific areas of the function concept, but there is no appears to be no universal panacea.

The idea of function as a process may prove to be a suitable cognitive root for the formal concept, but along the line of cognitive development there are obstacles to be overcome, including the encapsulation of the process as a single concept and the relating of this concept to its many and varied alternative representations. It remains a large and complex schema of ideas requiring a broad range of experience to grasp in any generality.



Facets and layers of a concept