SIS Piemonte

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Corso di Fondamenti della Matematica

Nodi fondamentali in Matematica

A. Sfard begins with a *process* acting on familiar objects which is

first *interiorized*,

Then *condensed* “in terms of input/output without necessarily considering its component steps”

and then *reified* as an “object-like entity”.

Action views of functions

In the context of an action/process/object theory of conceptual development, an action view involves an understanding of function as a non-permanent construct. An action view pertains to the computational aspects associated with functions, such as an arithmetic process or a ‘function machine’. For example, one can consider the function $f(x) = 3x^2 – 7$ to be an algorithm used to compute numeric values for a given input. This conception does not require an awareness of patterns and regularities that may exist between the numeric values of successive inputs and outputs, nor attention to causal and dependency relationships between inputs and outputs. An action conception is concerned with the computation of a single quantity for a single numeric value via a given algorithm or rule of association.

Process views of functions

Briedenbach et al. (1992) define a *process* conception of function as a complete understanding of a given transformational activity performed on a function, consisting of causal and dependency relationships which exist between the dependent and independent variables.

They claim that the process conception provides an entryway into an object-oriented understanding of function. Students are more able to comprehend properties such as 1–1, onto, and invertibility once a process conception is achieved.

Object-Oriented Views of Function

Sfard (1991, Sfard & Linchevski, 1994) describes an object-oriented conception as the *reification* of an action view. For example, one can consider the expression 3(x+5)+1 as an algorithm with which to produce various outputs. However, one can also “see” the expression as a certain number in its right; the expression becomes the result of the process.

Object-Oriented Views of Function

Student difficulties in simultaneously comprehending these meanings of an expression have been referred to as the ‘process-product dilemma’. Thinking of this type illustrates the beginnings of an object oriented view since the expression is considered to be a fixed value of an unknown. Hence, a single input-output action is conceived as a single entity. Generalizing this conception to involve the notion of variable, where the above expression simultaneously denotes several processes, represents thinking in line with functional algebra and is considered to be a structural conception. Sfard’s emphasis is on the development of ‘abstract objects’ as a product of a deeper understanding of a mathematical operation.

Gray & Tall describe a *procept* essentially as the amalgam of three things:

- a *process* (such as addition of three and four),
- a con*cept* produced by that process (the sum)
- a *symbol* that evokes either concept or process (e.g. 3+4).

Following Davis, they distinguish between a process which may be carried out by a variety of different algorithms and a procedure which is a "specific algorithm for implementing a process". A procedure is therefore cognitively more primitive than a process.

3° incontro

Le funzioni: storia ed epistemologia

The concept of function is one of the distinguishing features of "modern" as against "classical" mathematics. W. L. Schaaf goes a step further:
The keynote of Western culture is the function concept, a notion not even remotely hinted at by any earlier culture. And the function concept is anything but an extension or elaboration of previous number concepts—it is rather a complete emancipation from such notions.

Two mental images:

the geometric: expressed in the form of a curve;

the algebraic: expressed as a formula

- first finite
- later allowing infinitely many terms

Another dycothomy in conceptions:

- new "logical"
  ("abstract," "synthetic," "postulational")
- old "algebraic"
  ("concrete," "analytic," "constructive")

Euler does not define the term "analytic expression," but tries to give it meaning by explaining that admissible "analytic expressions" involve the four algebraic operations, roots, exponentials, logarithms, trigonometric functions, derivatives, and integrals.

The entire approach is algebraic. Not a single picture or drawing appears (in v. 1).

Expansions of functions in power series play a central role in this treatise.

The notion of function in explicit form did not emerge until the beginning of the 18th century.

The main reasons that the function concept did not emerge earlier were:

- lack of algebraic prerequisites
- lack of motivation.

In the course of about two hundred years (ca. 1450–1650), there occurred a number of developments that were fundamental to the rise of the function concept:

- Extension of the concept of number ;
- The creation of a symbolic algebra (Viète, Descartes, et al.);
- The study of motion as a central problem of science (Kepler, Galileo, …);
- The wedding of algebra and geometry (Fermat, Descartes,...).

The 17th century witnessed the emergence of modern mathematized science and the invention of analytic geometry.

A dynamic, continuous view of the functional relationship Vs/ the static, discrete view held by the ancients.

In the blending of algebra and geometry, the key elements were the introduction of

> *Variables* and the expression of the relationship between variables by means of
>
> *Equations*. The latter provided a large number of examples of curves

What was lacking was the identification of the independent and dependent variables in an equation.

The calculus developed by Newton and Leibniz had not the form that students see today.

In particular, it was not a calculus of *functions*. The principal objects of study in 17thcentury calculus were (geometric) curves. (For ex., the cycloid)

In fact, 17th-century analysis originated as a collection of methods for solving problems about curves: finding tangents to curves, areas under curves, lengths of curves, and velocities of points moving along curves.

Since the problems that gave rise to the calculus were geometric and kinematic in nature, time and reflection would be required before the calculus could be recast in algebraic form.

The variables associated with a curve were geometric—abscissas, ordinates, subtangents, subnormals, and the radii of curvature of a curve.

In 1692, Leibniz introduced the word "function" (see [25, p. 272]) to designate a geometric object associated with a curve.

For example, Leibniz asserted that "a tangent is a function of a curve" [12 p. 85].

1646-1716

Newton's "method of fluxions" applies to "fluents," not functions. Newton calls his variables "fluents"—the image (as in Leibniz) is geometric, of a point "flowing" along a curve.

Newton's major contribution to the development of the function concept was his use of power series. These were important for the subsequent development of that concept.

1643-1727

IL MOVIMENTO:

il punto di vista della fisica

Le radici del concetto di funzione si sono sviluppate quali relazioni tra variabili concrete, dinamiche e continue, per esprimere

l'idea di cambiamento

fenomeni di movimento

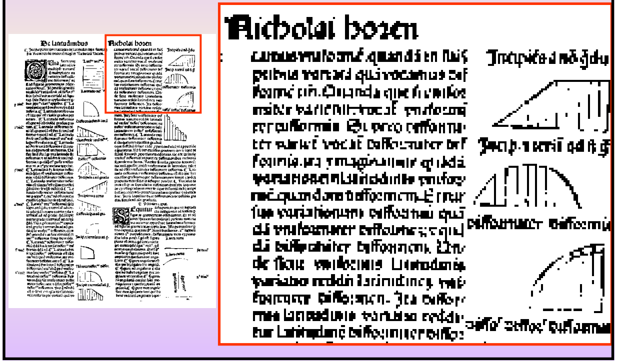
J. Kaput (1979), …

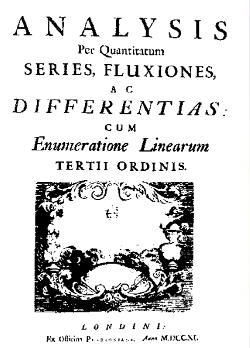
L'analisi della velocità

N. Oresme (Paris, 1325-1382)

Merton College (Oxford, XIV c.)

Tractatus de configurationibus qualitatum et motuum (1353)





ANALYSIS
Per Quantitatum
SERIES, FLUXIONES,
AC
DIFFERENTIAS:
CUM
Enumeratione Linearum
TERTII ORDINIS.

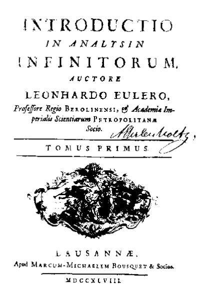
LONDINI:
Ex Officina Pearsoniana. *Anno* M.DCC.XI.

Newton:
la genesi attraverso il movimento

I …consider Mathematical Quantities …as generated by continual motion…
These Geneses are founded upon Nature and are every Day seen in the motion of Bodies.

[Newton, Quadr. Curves, 1710]

I GRAFICI:
il punto di vista della geometria e la progressiva de-geometrizzazione (da Eulero a Eulero)



INTRODUCTIO
IN ANALYSIN
INFINITORUM,
AUCTORE
LEONHARDO EULERO,
Profeſſore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ
Socio.

TOMUS PRIMUS.

LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLVIII.

"…Thus any function of x, geometrically interpreted… will correspond to a well defined line, straight or curve, the nature of which will depend on the nature of the function."

Bernoulli's definition of 1718:

One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants

This was the first formal definition of function, although Bernoulli did not explain what "composed in any manner whatever" meant.

1700-1782

This process of “degeometrization of analysis” saw the replacement of the concept of variable, applied to geometric objects, with the concept of function as an algebraic formula.

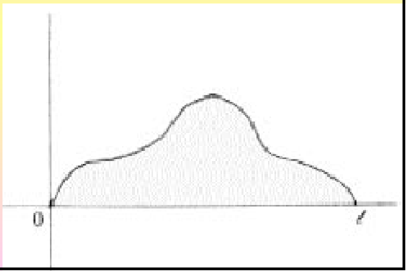
This trend was embodied in Euler’s classic *Introductio in Analysin Infinitorum* of 1748.

Euler begins by defining a function as an “analytic expression” (that is, a “formula”):

A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities.

The Vibrating-String Controversy.

An elastic string having fixed ends (0 and say) is deformed into some initial shape and then released to vibrate. The problem is to determine the function that describes the shape of the stringat time *t*.



An “article of faith”of 18th century mathematics:

If two analytic expressions agree on an interval, they agree everywhere.

(1717-1783)

In 1747, d’Alembert solved the Vibrating-String Problem by showing that the motion of the string is governed by the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

(a is a constant),

the so-called *wave equation*.

He solved this partial differential equation to obtainas the "most general" solution of the Vibrating-String Problem:

$$y(x,t) = [\phi(x + at) + \phi(x - at)]/2$$

being ϕ an "arbitrary" function.

ϕ is determined on (0,l) by the initial shape of the string, and is continued (by the "article of faith") as an odd periodic function of period 2l.

D'Alembert believed that the function ϕ (and hence f) must be an "analytic expression"—that is, it must be given by a formula.

In 1748, Euler wrote a paper on the same problem in which he agreed completely with d'Alembert concerning the solution but differed from him on its interpretation.

Euler claimed his experiments showed that the solution $y(x,t) = [\phi(x + at) + \phi(x - at)]/2$ gives the shapes of the string for different values of *t*, even when the initial shape is not given by a (single) formula.

From physical considerations, Euler argued that the initial shape of the string can be given

(a) by several analytic expressions in different subintervals of (say, circular arcs of different radii in different parts of or, more generally,

(b) by a curve drawn free-hand.

Daniel Bernoulli entered the picture in 1753 by giving yet another solution of the Vibrating-String Problem:

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \cos\frac{n\pi at}{\ell}.$$

This, of course, meant that an arbitrary function f(x) can be represented on (0,l) by a series of sines.

(Bernoulli was only interested in solving a physical problem, and did not give a definition of function. By an "arbitrary function" he meant an "arbitrary shape" of the vibrating string.)

Both Euler and d'Alembert (as well as other mathematicians of that time) found Bernoulli's solution absurd. Relying on the 18th century "article of faith," they argued that since f(x) and the sine series agree on (0.l) they must agree everywhere. But then one arrived at the manifestly absurd conclusion that an "arbitrary" function is odd and periodic.

Ravetz characterized the essence of the debate as one between d'Alembert's mathematical world, Bernoulli's physical world, and Euler's "no-man's land" between the two.

Euler's own view of functions evolved over a period of several years. See the following definition given in 1755:

If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others. If, therefore, x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions ...

Fourier
(1807, 1822)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right],$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\frac{n\pi t}{\ell}\,dt$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\frac{n\pi t}{\ell}\,dt.$$

(1768-1830)

Fourier, of course, claimed that it is true for *all* functions,where the term “function” was given the most general contemporary interpretation:

In general, the function represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there are an equal number of ordinates All have actual numerical values, either positive or negative or null. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity

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Fourier’s work raised the analytic (algebraic) expression of a function to at least an equal footing with its geometric representation (as a curve).

Fourier’s work:

- Did away with the “article of faith” held by 18th-century mathematicians.
- Showed that Euler’s concept of “discontinuous” was flawed.
- Gave renewed emphasis to analytic expressions.

In short:

0. Geometrical image: Leibniz, Euler (I)

1. An analytic expression (an “arbitrary” formula): Euler (II), DAlembert

2. then a curve (drawn freehand): Euler (III), Bernoulli

3. Then again an analytic expression (but this time a “specific” formula, namely a Fourier series): Fourier

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Dirichlet

In a fundamental paper of 1829, Dirichlet gave sufficient conditions for Fourier representability:

If a function f has only finitely many discontinuities and finitely many maxima and minima in (-l, l) then f may be represented by its Fourier series on (-l, l)

(The Fourier series converges pointwise to f where f is continuous, and to [f(x+) + f(x-)]/2 at each point x where f is discontinuous.)

(1805-1859)

For a mathematically rigorous proof of this theorem, one needed

(a) clear notions of continuity, convergence, and the definite integral, and

(b) clear understanding of the function concept.

Cauchy contributed to the former (*Cours d'Analyse,* 1821) and Dirichlet to the latter.

(1789-1857)

Dirichlet's definition of function:

y is a function of a variable x, defined on the interval if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established.

Dirichlet was the first to take seriously the notion of function as an arbitrary correspondence.
This is made abundantly clear in his 1829 paper on Fourier series, at the end of which he gives an example of a function (the *Dirichlet* function).

The Dirichlet function:

- was the first explicit example of a function that was not given by an analytic expression (or by several such), nor was it a curve drawn freehand;
- was the first example of a function that is discontinuous (in our, not Euler's sense) *everywhere*;
- illustrated the concept of function as an arbitrary pairing.

"Pathological" Functions.

Riemann (*Habilitationsschrift,* 1854):

Riemann extended Cauchy's concept of integral and thus enlarged the class of functions representable by Fourier series. This extension (known today as the Riemann integral) applies to functions of bounded variation, a much broader class of functions than Cauchy's continuous functions. Thus, a function can have infinitely many discontinuities (which can be dense in any interval) and still be Riemann-integrable.

In 1872, Weierstrass startled the mathematical community with his famous example of a continuous nowhere-differentiable function.

This example was contrary to all geometric intuition. In fact, up to about 1870, most books on the calculus "proved" that a continuous function is differentiable except possibly at a finite number of points! Even Cauchy believed that.

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x),$$

a is an odd integer, *b* a real number in and *ab* > 1 + 3p/2

(1826-1866)

Summary. Stimulated by Dirichlet's conception of function and his example the notion of function as an arbitrary correspondence is given free rein and gains general acceptance; the geometric view of function is given little consideration. (Riemann's and Weierstrass' functions could certainly not be "drawn," nor could most of the other examples given during this period.) After Dirichlet's work, the term "function" acquired a clear meaning independent of the term "analytic expression."

During the next half century, mathematicians introduced a large number of examples of functions in the spirit of Dirichlet's broad definition, and the time was ripe for an effortto determine which functions were actually describable by means of "analytic expressions", a vague term in use during the previous two centuries.

There was a famous exchange of letters in 1905 among Baire, Borel, Hadamard, and Lebesgue concerning the current logical state of mathematics.

Much of the debate was about function theory—the critical question being whether a definition of a mathematical object (say a number or a function), however given, legitimizes the existence of that object; in particular, whether Zermelo's axiom of choice is a legitimate mathematical tool for the definition or construction of functions.

Baire, Borel, and Lebesgue supported the requirement of a definite "law" of correspondence in the definition of a function. The "law," moreover, must be reasonably explicit—that is, understood by and communicable to anyone who wants to study the function (Borel thought experiment).

René-Louis Baire
(1874-1932)

Felix Borel
(1871-1956)

Henri Lebesgue
(1875-1941)

Jacques Hadamard
(1865-1963)

Ernst Zermelo
(1871-1953)

I fenomeni di cambiamento e di movimento possono produrre a risonanze cognitive positive negli allievi e supportare il loro apprendimento.

N. Bourbaki, *Éléments de mathématique*, Hermann, 1970

> Déf. 9. … une correspondance f = (F,A,B) est un fonction si, pour tout x appartenant à l'ensemble de départ A de f, la relation (x,y) ε F est fonctionelle en y; l'objet unique correspondant à x par f s'appelle la valeur de f pour l'élément x de A, et se désigne par f(x)… (II, Théorie des ensembles, §3, n°4)

Una corrispondenza f=(f,a,b) e' una funzione se, per ogni x appartenente all'insieme di partenza A di f, la relazione (x,y) e F e' funzionale in y; l'unico oggetto corrispondente a x tramite f si chiama valore di f per l'elemento x di A e si indica con f(x).

Però…

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