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Gelfand-Shilov Window Classes for Weighted Modulation Spaces

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We show that the Gelfand-Shilov algebra $\mathcal{S}_{1/2}^{1/2}$ is densely embedded in the weighted modulation space M_m^1. Here the weight function m is allowed to have a super-exponential growth at infinity. The basic tool is given by an integral transform called *short-time Fourier transform* (STFT). The STFT is used to both define and characterize the previous spaces. Moreover, our result is attained using the properties of the STFT and its adjoint.

Keywords: Ultra-distributions, Gelfand-Shilov type spaces, modulation spaces, short-time Fourier transform.

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1 Introduction

The aim of this paper is to study Gelfand-Shilov spaces as window classes for the weighted modulation spaces $M_m^{p,q}$, $1 \le p, q \le \infty$. The definition of modulation spaces is given by means of an integral transform coming from time-frequency analysis: the short-time Fourier transform (STFT). Indeed, we consider the linear operators of translation and modulation (so-called time-frequency shifts) given by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t) \,. \tag{1}$$

Next, let g be a non-zero window function in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, then the short-time Fourier transform (STFT) of a signal $f \in L^2(\mathbb{R}^d)$ with respect to the window g is given by

$$V_g f(x, \omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\, \overline{g(t - x)}\, e^{-2\pi i \omega t}\, dt \,. \tag{2}$$

We have $V_g f \in L^2(\mathbb{R}^{2d})$. This definition can be extended to pairs of dual topological vector spaces, whose duality, denoted by $\langle \cdot, \cdot \rangle$, extends the inner product on $L^2(\mathbb{R}^d)$.

Just a few words to explain the intuitive meaning of the previous “time-frequency” expression. If $f(t)$ represents a signal varying in time, its Fourier transform $\hat{f}(\omega)$ shows the distribution of its frequency ω, without any additional information about “when” these frequencies appear. To overcome this problem, one may choose a non-negative window function g well localized around the origin. Then, the information of the signal f at the instant x can be obtained by shifting the window g till the instant x under consideration, and by computing the Fourier transform of the product $f(x)g(t - x)$, that localizes f around the instant time x.

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Once the analysis of the signal f is terminated, we can reconstruct the original signal f by a suitable inversion procedure. Namely, the reproducing formula related to the STFT, for every pairs of windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ with $\langle \varphi_1, \varphi_2 \rangle \neq 0$, reads as follows

$$\int_{\mathbb{R}^{2d}} V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2 \, dx d\omega = \langle \varphi_2, \varphi_1 \rangle f \, . \tag{3}$$

The function φ_1 is called the *analysis* window, because the STFT $V_{\varphi_1} f$ gives the time-frequency distribution of the signal f, whereas the window φ_2 permits to come back to the original f and, consequently, is called the *synthesis* window. We address to Section 2 for a full treatment of the topic.

The modulation space norm $M^{p,q}_m(\mathbb{R}^d)$ of a function f on $\mathbb{R}^d$ is given by the $L^{p,q}_m(\mathbb{R}^{2d})$ norm of its STFT $V_g f$, defined on the time-frequency space $\mathbb{R}^{2d}$, with respect of a *suitable* window function g on $\mathbb{R}^d$. Depending on the growth of the weight function m, different Gelfand-Shilov classes may be chosen as fitting test function spaces for modulation spaces, see [5,15]. The widest class of weights allowing to define modulation spaces is the weight class $\mathcal{N}$. A weight function m on $\mathbb{R}^{2d}$ belongs to $\mathcal{N}$ if it is a continuous, positive function such that

$$m(z) = o(e^{cz^2}), \quad \text{for } |z| \to \infty, \quad \forall c > 0, \tag{4}$$

with $z \in \mathbb{R}^{2d}$. For instance, every function $m(z) = e^{s|z|^b}$, with $s > 0$ and $0 \leq b < 2$, is in $\mathcal{N}$. Thus, the weight m may grow faster than exponentially at infinity. We notice that there is a limit in enlarging the weight class for modulation spaces, imposed by Hardy's theorem: if $m(z) \geq Ce^{cz^2}$, for some $c > \pi/2$, then the corresponding modulation spaces are trivial [12].

Our main result (Theorem 3.3) shows that the Gelfand-Shilov class $\mathcal{S}^{1/2}_{1/2}$ is densely embedded in M^1_m, with $m \in \mathcal{N}$. Some applications of the previous result, restricted to a subclass $\mathcal{M}_v$ of $\mathcal{N}$, are contained in [4,5]. Namely, let v be a continuous, positive, even function ($v(z) = v(-z)$), with $v(0) = 1$. v is also assumed to be submultiplicative: $v(z_1+z_2) \leq v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$, and to satisfy: $v(x,\omega) = v(-\omega, x) = v(-x, \omega)$. Then $m \in \mathcal{M}_v$ (the class of v-moderate weights) if m is a positive, even continuous functions on $\mathbb{R}^{2d}$ that satisfies

$$m(z_1 + z_2) \leq C v(z_1) m(z_2) \quad \forall z_1, z_2 \in \mathbb{R}^{2d} \, .$$

Notice that the submultiplicativity and continuity of v imply that the weight v (and, consequently, every v-moderate weight m) is *dominated* by an exponential function, i.e.,

$$\exists\, C, k > 0 \quad \text{such that} \quad v(z) \leq C e^{k|z|}, \quad z \in \mathbb{R}^{2d}, \tag{5}$$

thereby $\mathcal{M}_v \subset \mathcal{N}$.

The results of [4,5] rely on the following property, proved in Section 3, by applying the above-mentioned density result:

Proposition 1.1 *For $m \in \mathcal{M}_v$, the space of admissible windows for computing the $M^{p,q}_m$-norm can be enlarged to M^1_v.*

This guarantees that every non-zero function $g \in M^1_v$ can be picked up to define the norms of the spaces $M^{p,q}_m$.

Notation. The scalar product on $\mathbb{R}^d$ is $xy := x \cdot y$. Hence, for every $c \in \mathbb{R}$, $cx^2 = c(x_1^2 + \cdots + x_d^2)$. Given a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, the partial derivative with respect to x_j is denoted by $\partial_j = \frac{\partial}{\partial x_j}$. Given a multi-index $p = (p_1, \ldots, p_d) \geq 0$, i.e., $p \in \mathbb{N}_0^d$ and $p_j \geq 0$, we write $\partial^p = \partial_1^{p_1} \cdots \partial_d^{p_d}$; moreover, we write $x^p = (x_1, \ldots, x_d)^{(p_1, \ldots, p_d)} = \prod_{i=1}^d x_i^{p_i}$. For $h \in \mathbb{R}$, $\alpha \in \mathbb{R}^d_+$, $h|x|^{1/\alpha} = h \sum_{i=1}^d |x_i|^{1/\alpha_i}$. Moreover, for $p \in \mathbb{N}_0^d$, we set $(p!)^\alpha = (p_1!)^{\alpha_1} \ldots (p_d!)^{\alpha_d}$, while as standard $p! = p_1! \ldots p_d!$.

The Schwartz class is denoted by $\mathcal{S}$, the space of tempered distributions by $\mathcal{S}'$. We use the brackets $\langle f, g\rangle$ to denote the extension of the inner product on $L^2(\mathbb{R}^d)$ to every pair of dual topological vector spaces. The space of smooth functions is $\mathcal{C}^\infty$ whilst the space of smooth functions with compact support is $\mathcal{C}_0^\infty$. The Fourier transform is normalized to be $\mathcal{F}f(\omega) = \int f(t)e^{-2\pi it\omega}dt$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous and dense embedding of the topological vector space B_1 into B_2.

2 Function Spaces

2.1 *Gelfand-Shilov Spaces.*

The Gelfand-Shilov spaces $\mathcal{S}_\beta^\alpha$ and Σ_β^α were first introduced by Gelfand and Shilov in [10]. They have been applied by many authors, both in time-frequency and in PDE contexts [2, 13–15]. We shall present them in the indices' case: $\alpha = \beta$, required by this study. Besides, instead of their very definition, we shall utilize the following useful characterization:

PROPOSITION 2.1 *The following statements are equivalent [5, 10, 13]:*
(i) $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ *(resp.* $f \in \Sigma_\alpha^\alpha(\mathbb{R}^d)$*).*
(ii) $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ *and there exist real constants* $h > 0, k > 0$ *such that (resp. for every* $h > 0, k > 0$*):*

$$\|fe^{h|x|^{1/\alpha}}\|_{L^\infty} < \infty \quad \textit{and} \quad \|(\mathcal{F}f)e^{k|\omega|^{1/\alpha}}\|_{L^\infty} < \infty. \tag{6}$$

(iii) $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ *and there exists* $C > 0$ *(resp. for every* $C > 0$*)*

$$\|x^p\partial^q f\|_{L^\infty} \leq RC^{|p|+|q|}(p!)^\alpha(q!)^\alpha, \quad \forall p, q \in \mathbb{N}_0^d, \tag{7}$$

for a suitable $R > 0$.

Gelfand-Shilov spaces enjoy the following embeddings:
(i) For $\alpha \geq 0$ [10],

$$\Sigma_\alpha^\alpha \hookrightarrow \mathcal{S}_\alpha^\alpha \hookrightarrow \mathcal{S}. \tag{8}$$

(ii) For every $0 \leq \alpha_1 < \alpha_2$ [5],

$$\mathcal{S}_{\alpha_1}^{\alpha_1} \hookrightarrow \Sigma_{\alpha_2}^{\alpha_2}. \tag{9}$$

Furthermore, S_α^α is not trivial if and only if $\alpha \geq 1/2$ whereas $\Sigma_\alpha^\alpha \neq \{0\}$ if and only if $\alpha > 1/2$ (see Pilipović [14]).

The strong duals of Gelfand-Shilov classes $\mathcal{S}_\alpha^\alpha$ and Σ_α^α are spaces of tempered ultra-distributions of Roumieu and Beurling type and will be denoted by $(\mathcal{S}_\alpha^\alpha)'$ and $(\Sigma_\alpha^\alpha)'$, respectively.

The Gelfand-Shilov spaces S_α^α are invariant under Fourier transform $\mathcal{F}$ and time-frequency shifts:

$$\mathcal{F}(S_\alpha^\alpha) = S_\alpha^\alpha, \quad T_x(\mathcal{S}_\alpha^\alpha) = \mathcal{S}_\alpha^\alpha \quad \text{and} \quad M_\omega(\mathcal{S}_\alpha^\alpha) = \mathcal{S}_\alpha^\alpha\,. \tag{10}$$

The same holds true for the spaces Σ_α^α. Therefore, the spaces S_α^α and Σ_α^α are a family of *Fourier transform and time-frequency shift invariant spaces* contained in the Schwartz class $\mathcal{S}$; the *smallest*, non-trivial one being $\mathcal{S}_{1/2}^{1/2}$. Functions in $\mathcal{S}_{1/2}^{1/2}$ are, e.g., the Gaussian $f(x) = e^{-\pi x^2}$ or the Hermite functions.

Another useful characterization of the spaces $\mathcal{S}_\alpha^\alpha$ and Σ_α^α involves the STFT and is proved in [13, Proposition 4.3]: $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ if and only if $V_g f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$, and the same for Σ_α^α. We shall heavily use the case $\alpha = 1/2$: for a non-zero window $g \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$ we have

$$f \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d) \Leftrightarrow V_g f \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^{2d}). \tag{11}$$

For a non-zero $g \in L^2(\mathbb{R}^d)$, we write V_g^* for the adjoint of V_g, given by

$$\langle V_g^* F, f\rangle = \langle F, V_g f\rangle, \quad f \in L^2(\mathbb{R}^d),\ F \in L^2(\mathbb{R}^{2d}).$$

In particular, for $F \in \mathcal{S}(\mathbb{R}^{2d})$, $g \in \mathcal{S}(\mathbb{R}^d)$, we have

$$V_g^* F(t) = \int_{\mathbb{R}^{2d}} F(x,\omega) M_\omega T_x g(t)\, dx\, d\omega \in \mathcal{S}(\mathbb{R}^d). \tag{12}$$

Take $f \in \mathcal{S}(\mathbb{R}^d)$ and set $F = V_g f$, then

$$f(t) = \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{R}^{2d}} V_g f(x,\omega) M_\omega T_x g(t)\, dx\, d\omega \in \mathcal{S}(\mathbb{R}^d). \tag{13}$$

We refer to [11, Proposition 11.3.2] for a whole treatment of the adjoint operator.

Lemma 2.2 *Consider a non-zero window $g \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$. Then, for every $F \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^{2d})$ we have $V_g^* F \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$.*

Proof We set $f = V_g^* F \in \mathcal{S}(\mathbb{R}^d)$ and we shall show that $f \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$ by using the characterization (7). To estimate $\|t^p \partial^q f\|_{L^\infty}$, we first recompute formula [11, Lemma 11.2.1] with the role of the operators ∂^q and t^p interchanged:

$$t^p \partial^q (M_\omega T_x g) = \sum_{\delta_1 \le q} \sum_{\delta_2 \le p} \binom{q}{\delta_1} \binom{p}{\delta_2} x^{\delta_2} (2\pi i \omega)^{\delta_1} M_\omega T_x (t^{p-\delta_2} \partial^{q-\delta_1} g). \tag{14}$$

Secondly, for $\delta_1 \le q$ and $\delta_2 \le p$, we get the following majorization:

$$\begin{aligned} \|M_\omega T_x (t^{p-\delta_2} \partial^{q-\delta_1} g)\|_{L^\infty} &= \|t^{p-\delta_2} \partial^{q-\delta_1} g\|_{L^\infty} \\ &\le C^{|p-\delta_2|+|q-\delta_1|+1} ((p-\delta_2)!)^{1/2} ((q-\delta_1)!)^{1/2} \end{aligned}$$

The preceding estimates yield the desired result. In fact,

$$\begin{aligned} |t^p \partial^q f| &\le \int_{\mathbb{R}^{2d}} |F(x,\omega)|\, |x^p \partial^q (M_\omega T_x g)|\, dx\, d\omega \\ &\le \sum_{\delta_1 \le q} \sum_{\delta_2 \le p} \binom{q}{\delta_1} \binom{p}{\delta_2} C^{|p-\delta_2|+|q-\delta_1|+1} ((p-\delta_2)!)^{1/2} ((q-\delta_1)!)^{1/2} \\ &\qquad \cdot \int_{\mathbb{R}^{2d}} |F(x,\omega)|\, |x^{\delta_2} (2\pi i \omega)^{\delta_1}|\, dx\, d\omega. \end{aligned}$$

In view of our assumption $F \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^{2d})$, by using the characterization (7), the last integral can be easily estimated by $\tilde{C}^{|\delta_2|+|\delta_1|+1} (\delta_2!)^{1/2} (\delta_1!)^{1/2}$, hence $\|t^p \partial^q f\|_{L^\infty} \le K^{|p|+|q|+1} (p!)^{1/2} (q!)^{1/2}$, for every $p, q \in \mathbb{N}_0^d$. □

2.2 *Modulation Spaces.*

Modulation spaces having weights in $\mathcal{M}_v$ (at most sub-exponential growth) were first introduced by Feichtinger in [6]. We define them using the Gelfand-Shilov class $\mathcal{S}^{1/2}_{1/2}$ as test function space in the way hereafter. Let us first argue for weights in the more general class $\mathcal{N}$.

Definition 2.3 Let $m \in \mathcal{N}$, and g a non-zero *window* function in $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$. For $1 \le p, q \le \infty$, the *modulation space* $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered ultra-distributions $f \in (\mathcal{S}_{1/2}^{1/2})'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M_m^{p,q}$ is

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{1/q},$$

(with obvious changes if either $p = \infty$ or $q = \infty$). If $p = q$, $M_m^p := M_m^{p,p}$, and, if $m \equiv 1$, then $M^{p,q}$ and M^p stand for $M_m^{p,q}$ and $M_m^{p,p}$, respectively.

Notice that:
(i) If $f, g \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$, the above integral is convergent thanks to (4), (6) and (11). Namely, if $m \in \mathcal{N}$ we choose $c = h - \epsilon > 0$ in (4), for a suitable $\epsilon > 0$, with h being such that $\|V_g f e^{h|\cdot|^2}\|_{L^\infty} < \infty$. Hence

$$\begin{aligned} &\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \\ &\le C \, \|(V_g f) e^{h|\cdot|^2}\|_{L^\infty} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |m(x,\omega)|^p e^{-hp|(x,\omega)|^2} \, dx \right)^{q/p} d\omega < \infty. \end{aligned}$$

(ii) The definition of $M_m^{p,q}$ may depend on the choice of the window function g. However, we shall see in the next section that, if $m \in \mathcal{M}_v$, the definition of $M_m^{p,q}$ does not depend on $g \in \mathcal{S}_{1/2}^{1/2}$: actually, the class of admissible window can be enlarged to M_v^1. Since the previous environment is the main concern of this paper, we shall continue to denote $M_m^{p,q}$ for every $m \in \mathcal{N}$, keeping in mind that, if $m \in \mathcal{N} \setminus \mathcal{M}_v$, we have $M_m^{p,q} = M_{m,g}^{p,q}$, where $g \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$ is the window with respect we compute the STFT V_g.
(iii) For $m \in \mathcal{M}_v$, then $M_m^{p,q}$ is the subspace of ultra-distribution $(\Sigma_1^1)'$ defined in [5, Definition 2.1].
(iv) If $m \in \mathcal{M}_v$ and fulfills the GRS-condition $\lim_{n\to\infty} v(nz)^{1/n} = 1$, for all $z \in \mathbb{R}^{2d}$, the definition of modulation spaces is the same as in [3] (because the “space of special windows” $\mathcal{S}_{\mathcal{C}}$ is a subset of $\mathcal{S}_{1/2}^{1/2}$).
(v) For related constructions of modulation spaces, involving the theory of coorbit spaces, we refer to [8,9].

3 The Density Result

First, let us recall the following semigroup property of the Gaussian functions.

Lemma 3.1 *Let $a, b > 0$, then*

$$e^{-\pi a x^2} * e^{-\pi b x^2} = (a+b)^{-d/2} e^{-\pi ab\, x^2/(a+b)}. \tag{15}$$

Proof It is a straightforward computation by Fourier transform. Namely, recall that, for all $a > 0$, the Fourier transform of a Gaussian is a Gaussian again, given by $\mathcal{F}(e^{-\pi a x^2})(\omega) = a^{-d/2} e^{-\pi \omega^2/a}$. Therefore, we have

$$\begin{aligned} e^{-\pi a x^2} * e^{-\pi b x^2} &= \mathcal{F}^{-1}(\mathcal{F}(e^{-\pi a x^2}) \cdot \mathcal{F}(e^{-\pi b x^2})) \\ &= \mathcal{F}^{-1}(a^{-d/2} e^{-\pi \omega^2/a} \cdot b^{-d/2} e^{-\pi \omega^2/b}) \\ &= (ab)^{-d/2} \mathcal{F}^{-1}(e^{-\pi (a+b)\omega^2/(ab)}) = (a+b)^{-d/2} e^{-\pi ab\, x^2/(a+b)}. \end{aligned}$$

□

Let $\varphi(z) = e^{-\pi z^2}$, $z \in \mathbb{R}^{2d}$. The family $\{\varphi_n\}_n$, with n positive integer given by

$$\varphi_n(z) = n^{2d}\varphi(nz) \tag{16}$$

is an *approximate identity* and, by the characterization (11), is straightforward to show that $\varphi_n \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^{2d})$, for every $n \in \mathcal{N}$.

LEMMA 3.2 *Let $m \in \mathcal{N}$ and fix a window function $g \in \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$. Consider a function $f \in M^1_m(\mathbb{R}^d)$. Then, for every $l, n \in \mathbb{N}$, $l, n > 0$, the functions defined by*

$$G_{n,l}(z) = [(V_g f)m * \varphi_n)]\,(z)e^{-\pi z^2/l}, \quad z \in \mathbb{R}^{2d}, \tag{17}$$

are in $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^{2d})$.

Proof We use the characterization in (6). We consider $0 < h < \pi/l$. Then, using Young's inequality for the convolution norm estimate, we have

$$\begin{aligned}
\|G_{n,l}(z)e^{hz^2}\|_{L^\infty} &= \|\,[(V_g f)m * \varphi_n)]\,(z)e^{-\pi z^2/l}e^{hz^2}\|_{L^\infty}\\
&= \|(V_g f)m * \varphi_n)\|_{L^\infty}\|e^{-\pi z^2/l}e^{hz^2}\|_{L^\infty}\\
&\le \|(V_g f)m\|_{L^\infty}\|\varphi_n\|_{L^1}\|e^{-(\pi/l-h)z^2}\|_{L^\infty}\\
&= \|(V_g f)m\|_{L^\infty}\|e^{-(\pi/l-h)z^2}\|_{L^\infty} < \infty,
\end{aligned}$$

since $(V_g f)m \in \mathcal{C}_0$. Hence, the left-hand side inequality in (6) is fulfilled. Next, we compute the Fourier transform of $G_{n,l}$. Precisely,

$$\begin{aligned}
\mathcal{F}(G_{n,l})(\zeta) &= (\mathcal{F}\,[(V_g f)m * \varphi_n] * \mathcal{F}(e^{-z^2/l}))(\zeta)\\
&= (\mathcal{F}[(V_g f)m]\mathcal{F}(\varphi_n)) * (l^d e^{-\pi l\zeta^2})\\
&= l^d(\mathcal{F}[(V_g f)m]e^{-\pi\zeta^2/n^2}) * e^{-\pi l\zeta^2}.
\end{aligned}$$

We look for a positive integer k that satisfies the right-hand side inequality of (6). We use (15) to compute $e^{-\pi\zeta^2/n^2} * e^{-\pi l\zeta^2} = (1/n^2+l)^{-d}e^{-\pi l\zeta^2/(1+ln^2)}$. The boundedness of the Fourier transform between the spaces L^1 and L^∞ then gives the desired result:

$$\begin{aligned}
\|\mathcal{F}(G_{n,l})(\zeta)e^{k\zeta^2}\|_{L^\infty} &\le l^d\,\|\mathcal{F}[(V_g f)m]\|_{L^\infty}\|(e^{-\pi\zeta^2/n^2} * e^{-\pi l\zeta^2})e^{k\zeta^2}\|_{L^\infty}\\
&\le \left(\frac{l\,n^2}{1+l\,n^2}\right)^d \|(V_g f)m\|_{L^1}\|e^{-[\pi l/(1+l\,n^2)-k]\zeta^2}\|_{L^\infty}\\
&< \infty,
\end{aligned}$$

for every k such that $0 < k < \pi l/(1+l\,n^2)$. □

Eventually, we have all the pieces in place to prove our main result.

THEOREM 3.3 *Let $m \in \mathcal{N}$ and fix a non-zero window $g \in \mathcal{S}^{1/2}_{1/2}$ to define the modulation space M^1_m. Then,*

$$\mathcal{S}^{1/2}_{1/2} \hookrightarrow M^1_m.$$

Proof For sake of simplicity, we assume that the window g is normalized, that is $\|g\|_{L^2} = 1$. Let $f \in M^1_m(\mathbb{R}^d)$, where its M^1_m-norm is computed with respect to the fixed window g. Clearly, as pointed out after the definition of the modulation spaces $M_m^{p,q}$, we have $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d) \subset M_m^1(\mathbb{R}^d)$. We shall prove the density of the preceding embedding. Consider the sequence of functions $f_{n,l}$ defined as

$$f_{n,l} := V_g^* G_{n,l},$$

where the functions $G_{n,l}$ are given by (17). Using Lemmata 3.2 and 2.2 we obtain that $f_{n,l} \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$.

Since $(V_g f)m \in L^1$, the approximate identity $\{\varphi_n\}_n$ defined in (16) gives the L^1-convergence of the sequence $(V_g f)m * \varphi_n$ to the function $(V_g f)m$.

Next, observe that for every fixed l the functions $G_{n,l}$ converge to $(V_g f)me^{-\pi z^2/l}$ in L^1-norm, since

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2d}} |(V_g f)m * \varphi_n - (V_g f)m| e^{-\pi z^2/l}\, dz \le \|(V_g f)m * \varphi_n - (V_g f)m\|_{L^1}$$

and the right-hand side of the previous inequality tends to 0, when $n \to \infty$. Now, the Dominated Convergence Theorem gives that the sequence $(V_g f)me^{-\pi z^2/l}$, both dominated by and pointwise converging to the integrable function $(V_g f)m$, converges in the L^1-norm to $(V_g f)m$.

Finally, the inversion formula (13) proves that the sequence $f_{n,l}$ converges to f in the M_m^1-norm. In fact,

$$\begin{aligned} \|f_{n,l} - f\|_{M_m^1} &= \|[V_g(f_{n,l} - f)]m\|_{L^1} = \|G_{n,l} - (V_g f)m\|_{L^1} \\ &\le \|G_{n,l} - (V_g f)me^{-\pi z^2 l}\|_{L^1} + \|(V_g f)me^{-\pi z^2 l} - (V_g f)m\|_{L^1} \end{aligned}$$

and the right-hand side above tends to 0 for $(n,l) \to +\infty$, independently on the indices' order. □

Finally, we restrict our attention to v-moderate weights $m \in \mathcal{M}_v$. Then, as shown in [5], modulation spaces can be regarded as subspace of the space of ultra-distributions $(\Sigma_1^1)'$. In this case, in the previous paper is highlighted that they gain many nice properties. In particular, the definition of modulation spaces is independent of the choice of the non-zero window $g \in \Sigma_1^1$. Actually, thanks to our density result, the window class Σ_1^1 can be enlarged to M_v^1.

Proof of Proposition 1.1. We first notice that $\mathcal{S}_{1/2}^{1/2} \hookrightarrow \Sigma_1^1$, by means of the continuous and dense embedding given in (8). Finally, the inclusion $\Sigma_1^1 \subset M_v^1$ [5] yields to the dense embedding $\Sigma_1^1 \hookrightarrow M_v^1$ whilst the window class extension from Σ_1^1 to M_v^1 follows by the same arguments as in [11, Theorem 11.3.7]. □

For weights $m \in \mathcal{N} \setminus \mathcal{M}_v$, we lose the submultiplicative property and the standard techniques via Young's inequality can not be applied anymore. It seems reasonable to get the independence of the modulation space definition for windows g at least in $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$, but a new approach coming from atomic decomposition theory seems to be more appropriate and will be the object of a forthcoming work.

Since, for $m \in \mathcal{M}_v$, $M_m^1 \hookrightarrow M_m^{p,q}$, $1 \le p, q < \infty$, see [5,11], we have:

Corollary 3.4 *Let $m \in \mathcal{M}_v$, $1 \le p, q < \infty$, then*

$$\mathcal{S}_{1/2}^{1/2} \hookrightarrow M_m^{p,q}.$$

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