LOCALIZATION OF FRAMES II

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Abstract. The theory of localized frames is refined to include quasi-Banach spaces and spaces with multiple generators. Applications are given to non-linear approximation with frames and to the convergence of the iterative frame algorithm in finer norms, and to the characterization of Besov spaces with wavelet frames.

1. Introduction

The rise of frame theory in applied mathematics is due to the flexibility and reduncancy of frames. Structured frames are much easier to construct than structured orthonormal bases. Redundancy is useful in the case of lossy data, often provides more numerical stability, and plays a pivot role in the spectacular applications of $\Sigma\Delta$-quantization [12]. The concept of frames was introduced by Duffin and Schaeffer [15] and popularized greatly by the work of Daubechies and her coauthors [7, 8, 9, 10]. For up-to-date treatments of frames one may consult the monographs [4, 5, 29] or the special issue [2]. Although the frame concept is a pure Hilbert space concept, it was recognized early on that the usefulness of frames goes far beyond Hilbert space theory. For instance, wavelet frames are used to detect smoothness and compression properties of functions, Gabor frames characterize the time-frequency concentration of distributions, and frames of reproducing kernels yield sampling theorems in function spaces. In all these applications, the frames satisfy some additional hypotheses: the wavelet of a wavelet frame must satisfy smoothness and moment conditions, whereas the Haar wavelet is not appropriate; a Gaussian window works very well for Gabor frames, whereas a rectangular Gabor window does not capture frequency concentration. While in these concrete examples it is well understood what constitutes a "good frame," it is not obvious how to formulate appropriate conditions for arbitrary frames.

The preceding paper [27] is an attempt to understand "good frames". In our opinion, the relevant property to distinguish "good frames" among arbitrary frames is a new form of localization for frames. The main result of [27] asserts that the localization properties of a frame are inherited by its dual frame. For "good frames" in this sense, we show that a class of associated Banach spaces can be characterized completely by the magnitude of the frame coefficients and that the frame expansion is stable in these Banach spaces. The examples considered in [27] are the sampling

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theory in shift-invariant spaces and Gabor frames. Frames of reproducing kernels in shift-invariant spaces can be shown to be "local" and lead to local reconstruction formulas. Likewise, Gabor frames with suitable time-frequency localization possess a dual frame with the same time-frequency localization, and as a consequence the time-frequency concentration of distributions can be characterized using Gabor frames. These examples settled two conjectures about sampling and about Gabor frames in the literature. Another application of localized frames was given in [26] where a conjecture of Feichtinger was partially solved. The circulation of the preprint [27] has stimulated a number of further investigations: in [6] precise error estimates are derived for the finite section method, these work exactly for localized frames, in [16] an equivalence relation of localized frames and applications to α-modulation spaces are studied.

Independently and in a completely different direction, Balan, Casazza, Heil, and Landau [3] have introduced a similar concept of localization to define the density of an abstract frame. Based on their theory of localization, they prove deep inequalities relating this density to the excess of frames.

In this paper we study several new facets of localized frames and answer several questions that came up in discussions of localized frames. While our methods are mostly standard, we use Banach algebra techniques for some decisive arguments. Our results owe their depth to a highly non-trivial theorem of Jaffard and Journé [30].

Specifically, we will extend the characterization of functional properties by means of frames to certain quasi-Banach spaces. Quasi-Banach spaces have made a comeback through the rise of non-linear approximation theory. We will show that localized frames yield sparse representations for the associated classes of quasi-Banach spaces.

Next we will investigate the convergence of the iterative frame algorithm of Duffin and Schaeffer (and some of its accelerations) in the norm of the associated Banach spaces. Sometimes, in particular for frames without structure, the iterative frame algorithm is still preferred to more sophisticated methods, because it does not require any knowledge of the dual frame. We show that the iterative frame algorithm converges automatically in much finer norms. Our result shows that localized frames possess a surprising robustness in their numerical performance. To say it more flamboyantly, localized frames are "good frames" for numerical analysis.

As an example we will treat the sampling problem in shift-invariant spaces with multiple generators. This requires the study of frames that are localized with respect to Riesz basis with additional structure. In the final example we will characterize certain Besov spaces by localized wavelet frames. This characterization is remarkable in that we do not need any a priori assumptions on the dual frame. (A more extensive treatment of wavelet frames will be carried out elsewhere.)

The results presented here could be summarized by saying that localized frames are indeed "good frames". We hope that this concept will have many more applications and motivate further results.

The paper is organized as follows: In Section 2 we introduce the concept of localized frames and discuss the associated (quasi)-Banach spaces. In Section 3 we derive frame expansions and characterizations for the quasi-Banach spaces with parameter $p < 1$. Section 4 is devoted to non-linear approximation with frames and shows that the associated Banach spaces possess sparse representations with respect to localized frames. In Section 5 we investigate the convergence of the iterative frame algorithm. In Section 6 we adjust the concept of localization to deal with shift-invariant spaces with multiple generators. Finally, in Section 7 we apply localized wavelet frames for a characterization of certain Besov spaces.

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2. Localization of Frames and Associated Banach Spaces

We first review the new concept of localization for frames and summarize the main results from [27].

Throughout the paper $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ denotes for frame for a given Hilbert space $\mathcal{H}$; this means that the frame operator $Sf = S_{\mathcal{E}} f = \sum_{x\in\mathcal{X}} \langle f, e_x\rangle e_x$ is boundedly invertible on $\mathcal{H}$. The set $\{g_n : n \in \mathcal{N}\}$ is always a Riesz basis for $\mathcal{H}$ with dual basis $\{\widetilde{g_n} : n \in \mathcal{N}\}$. The set $\mathcal{N}$ is assumed to be separated in $\mathbb{R}^d$, i.e., $\inf_{x,y\in\mathcal{X}:x\neq y} |x-y| \geq \delta > 0$, and $\mathcal{X}$ is relatively separated, i.e., $\mathcal{X}$ is a finite union of separated sets. We should think of the index as a kind of localization. For instance, if $\mathcal{H} = L^2(\mathbb{R}^d)$, then the index x in $e_x(t)$ indicates that the essential support of e_x is centered at x.

2.1. **Localized Frames.**

Definition 1. A frame $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ for a Hilbert space $\mathcal{H}$ is called polynomially localized with respect to the Riesz basis $\{g_n\}$ with decay $s > 0$ (or simply *s-localized*), if

$$|\langle e_x, g_n\rangle| \leq C(1+|x-n|)^{-s}\,, \tag{1}$$

and

$$|\langle e_x, \widetilde{g_n}\rangle| \leq C(1+|x-n|)^{-s} \tag{2}$$

for all $n \in \mathcal{N}$ and $x \in \mathcal{X}$. Similarly, a frame $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ for $\mathcal{H}$ is called exponentially localized with respect to the Riesz basis $\{g_n\}$, if for some $\alpha > 0$

$$\max\{|\langle e_x, g_n\rangle|, |\langle e_x, \widetilde{g_n}\rangle|\} \leq Ce^{-\alpha|x-n|} \qquad \forall x \in \mathcal{X}, n \in \mathcal{N}\,. \tag{3}$$

The main theorem in the theory of localized frames asserts that the dual frame $\widetilde{\mathcal{E}} = \{\widetilde{e_x} := S^{-1}e_x : x \in \mathcal{X}\}$ possesses the same localization properties. In the following d is the dimension of the "carrier" space $\mathbb{R}^d$ for $\mathcal{N}$ and $\mathcal{X}$.

Theorem 2.1. *(a) If $\mathcal{E}$ is s-localized with respect to the Riesz basis $\{g_n\}$ for some $s > d$, then $\widetilde{\mathcal{E}}$ is also s-localized.*

(b) If $\mathcal{E}$ is exponentially localized, then $\widetilde{\mathcal{E}}$ is also exponentially localized (with a possibly different exponent in (3)*).*

To give the reader an idea what is involved in this statement, we sketch a short proof for polynomially localized frames. The precise details can be found in [27].

Proof. The proof is based on two important facts from the theory of Banach algebras. Let $\mathcal{A}_s$ be the class of $\mathcal{N}\times\mathcal{N}$-matrices $A=(a_{kl})$ such that

$$|a_{kl}| \le C(1+|k-l|)^{-s} \qquad \text{for all } k,l\in\mathcal{N}\,. \tag{4}$$

Jaffard [30] proved the following properties: (a) If $s>d$, then $\mathcal{A}_s$ is an algebra (under matrix multiplication). (b) If $A\in\mathcal{A}_s$ for $s>d$ and if A is invertible on $\ell^2(\mathcal{N})$, then $A^{-1}\in\mathcal{A}_s$. See [30], also [28] for a new proof and refinements.

Step 1. Let $(\Gamma f)(n)=\langle f,\widetilde{g_n}\rangle$. Since $\{g_n : k\in\mathcal{N}\}$ is a Riesz basis of $\mathcal{H}$, Γ is an isomorphism from $\mathcal{H}$ onto $\ell^2(\mathcal{N})$ with inverse $\Gamma^{-1}\mathbf{c}=\sum_{k\in\mathcal{N}}c_n g_n$ for $\mathbf{c}=(c_n)\in\ell^2(\mathcal{N})$. Then the frame operator S can be factored as

$$S=\Gamma^{-1}T\Gamma\,, \tag{5}$$

where T is the matrix of S with respect to the given Riesz basis. It has the entries

$$T_{kl}=\langle Sg_l,\widetilde{g_k}\rangle = \sum_{x\in\mathcal{X}}\langle g_l,e_x\rangle\langle e_x,\widetilde{g_k}\rangle \qquad k,l\in\mathcal{N}\,.$$

It follows that T is invertible on $\ell^2(\mathcal{N})$ if and only if S is invertible on $\mathcal{H}$.

Step 2. Let $b_{kl}=\max_{x\in\mathcal{X}\cap(k+[0,1]^d)}|\langle e_x,g_l\rangle|$ and $c_{kl}=\max_{x\in\mathcal{X}\cap(k+[0,1]^d)}|\langle e_x,\widetilde{g_l}\rangle|$ for $k,l\in\mathcal{N}$. The s-localization of $\mathcal{E}$ implies that $B,C\in\mathcal{A}_s$. Since $\mathcal{X}$ is relatively separated, $\sup_{k\in\mathcal{N}}$ card $\{x\in\mathcal{X} : x\in k+[0,1]^d\}=\nu<\infty$ and therefore $\sum_{x\in\mathcal{X}\cap(k+[0,1]^d)}|\langle e_x,g_l\rangle|\le \nu\, b_{kl}$ and similarly for C.

Step 3. We estimate the entries of T:

$$\begin{aligned}|T_{kl}| &\le \sum_{m\in\mathcal{N}}\sum_{x\in\mathcal{X}\cap(m+[0,1]^d)}|\langle g_l,e_x\rangle\,\langle e_x,\widetilde{g_k}\rangle| \\ &\le \nu^2\sum_{m\in\mathcal{N}} b_{ml}c_{mk}=(C^*B)_{kl}\,.\end{aligned}$$

Since B,C and C^* are all in the algebra $\mathcal{A}_s$, we find that $T\in\mathcal{A}_s$ as well, explicitly, $|T_{kl}|\le C(1+|k-l|)^{-s}, k,l\in\mathcal{N}$.

Step 4. Since $T\in\mathcal{A}_s$ and T is invertible on $\ell^2(\mathcal{N})$ by Step 1, Jaffard's Lemma implies that $T^{-1}\in\mathcal{A}_s$. Let U be the matrix with entries $u_{kl}=|(T^{-1})_{kl}|$, then also $U\in\mathcal{A}_s$.

Step 5. To show that $\widetilde{\mathcal{E}}$ is s-localized, we must check the size of $\langle S^{-1}e_x, g_l\rangle$ and $\langle S^{-1}e_x, \widetilde{g_n}\rangle$. Using (5) in the form $\Gamma S^{-1} = T^{-1}\Gamma$, we obtain that

$$\begin{aligned}
\widetilde{c}_{kl} &= \max_{x\in\mathcal{X}\cap(k+[0,1]^d)} |\langle S^{-1}e_x, \widetilde{g_l}\rangle| \\
&= \max_{x\in\mathcal{X}\cap(k+[0,1]^d)} |(\Gamma S^{-1}e_x)(l)| \\
&= \max_{x\in\mathcal{X}\cap(k+[0,1]^d)} |(T^{-1}\Gamma e_x)(l)| \\
&= \max_{x\in\mathcal{X}\cap(k+[0,1]^d)} |\sum_{m\in\mathcal{N}} (T^{-1})_{lm}\langle e_x, \widetilde{g_m}\rangle| \\
&\le \nu \sum_{m\in\mathcal{N}} u_{lm} \max_{x\in\mathcal{X}\cap(k+[0,1]^d)} |\langle e_x, \widetilde{g_m}\rangle| \\
&= \nu \sum_{m\in\mathcal{N}} u_{lm}c_{km} = \nu(CU^*)_{kl}\,.
\end{aligned}$$

Since $C, U \in \mathcal{A}_s$ and $\mathcal{A}_s$ is an algebra, we find that $\tilde{C} = (\tilde{c}_{kl}) \in \mathcal{A}_s$.

Step 6. To obtain the estimate for $\langle S^{-1}e_x, g_l\rangle$, we interchange the role of g_n and $\tilde{g_n}$. Precisely, define $\widetilde{\Gamma}f(n) = \langle f, g_n\rangle$ and $\widetilde{T}_{kl} = \langle S\widetilde{g_l}, g_k\rangle$, then $S = \widetilde{\Gamma}^{-1}\widetilde{T}\widetilde{\Gamma}$. The same argument as above shows that the matrix $\tilde{B}$ with entries $\tilde{b}_{kl} = \max_{x\in\mathcal{X}\cap(k+[0,1]^d)} |\langle S^{-1}e_x, g_l\rangle|$ is also in $\mathcal{A}_s$. This means that $\widetilde{\mathcal{E}}$ is s-localized. ■

For applications, including the solution of two conjectures in sampling theory and Gabor theory, the reader should consult [27], for a refinement of Theorem 2.1 to other decay conditions see [28].

2.2. **Associated Banach Spaces.** Let m be a positive, even, and continuous function on $\mathbb{R}^d$ and $\ell^p_m(\mathcal{X})$ the corresponding weighted ℓ^p-space defined by the norm

$$\|\mathbf{c}\|_{\ell^p_m} = \Big(\sum_{x\in\mathcal{X}} |c_x|^p m(x)^p\Big)^{1/p}, \tag{6}$$

with the usual modification for $p = \infty$.

Definition 2. Let $0 < p \le \infty$. If $\ell^p_m(\mathcal{N}) \subseteq \ell^2(\mathcal{N})$, then the Banach space $\mathcal{H}^p_m$ is defined to be

$$\mathcal{H}^p_m = \{f \in \mathcal{H} : f = \sum_{n\in\mathcal{N}} c_n g_n \text{ for } \mathbf{c} \in \ell^p_m(\mathcal{N})\} \tag{7}$$

with norm $\|f\|_{\mathcal{H}^p_m} = \|\mathbf{c}\|_{\ell^p_m}$. If $\ell^p_m \not\subseteq \ell^2(\mathcal{N})$ and $p < \infty$, then $\mathcal{H}^p_m$ is defined as the completion of the subspace $\mathcal{H}_0$ of finite linear combinations, i.e., $\mathcal{H}_0 = \{f = \sum_{n\in\mathcal{N}} c_n g_n : \operatorname{supp}\mathbf{c} \text{ finite}\}$, with respect to the norm $\|f\|_{\mathcal{H}^p_m} = \|\mathbf{c}\|_{\ell^p_m}$. If $p = \infty$ and $\ell^\infty_m \not\subseteq \ell^2$, then $\mathcal{H}^\infty_m$ is weak-* completion of $\mathcal{H}_0$.

REMARKS: 1. Note that c_n is uniquely determined, in fact, $c_n = \langle f, \widetilde{g_n}\rangle$ or $\mathbf{c} = \Gamma f$.

2. If $\ell^p_m(\mathcal{N}) \subseteq \ell^2(\mathcal{N})$, then $\mathcal{H}^p_m$ is a (dense) subspace of $\mathcal{H}$.

To make the analysis of the associated spaces accessible to harmonic analysis methods, we will only use two types of weights: (a) A non-negative, continuous, and even function m on $\mathbb{R}^d$ is called an *s-moderate* weight, if there are constants $C, s \geq 0$ such that

$$m(t+x) \leq C(1+|t|)^s m(x) \qquad \text{for all } t, x \in \mathbb{R}^d\,. \tag{8}$$

A weight function m is called subexponential if there are constants $C, \gamma > 0$ and $0 \leq \beta < 1$ such that

$$m(t+x) \leq C e^{\gamma |t|^\beta} m(x) \qquad \text{for all } t, x \in \mathbb{R}^d\,. \tag{9}$$

By setting $x = 0$ in (8) and in (9) we see that a an s-moderate weight m grows at most polynomially, i.e., $m(t) \leq C(1+|t|)^s$, and a subexponential weight grows at most like $Ce^{\gamma|t|^\beta}$.

2.3. **Frame Analysis of the Associated Banach Spaces.** A thorough analysis of the coefficient map and synthesis operator that are associated with every frame leads to a complete understanding of the Banach spaces $\mathcal{H}^p_m$ by means of their frame coefficients [27].

Theorem 2.2. *Assume that $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ is an r-localized frame with respect to the Riesz basis $\{g_n\}$ and that $r > s + d$.*

(a) Then the frame operator S is invertible simultaneously on all Banach spaces $\mathcal{H}^p_m$, where $1 \leq p \leq \infty$ and all s-moderate weights m.

(b) The frame expansion

$$f = \sum_{x\in\mathcal{X}} \langle f, e_x\rangle \widetilde{e_x} = \sum_{x\in\mathcal{X}} \langle f, \widetilde{e_x}\rangle e_x \tag{10}$$

converges unconditionally in $\mathcal{H}^p_m$ for $1 \leq p < \infty$ (and weak unconditionally in $\mathcal{H}^\infty_m$).*

(c) We have the norm equivalence

$$\|f\|_{\mathcal{H}^p_m} \asymp \Big(\sum_{x\in\mathcal{X}} |\langle f, e_x\rangle|^p m(x)^p\Big)^{1/p} \asymp \Big(\sum_{x\in\mathcal{X}} |\langle f, \widetilde{e_x}\rangle|^p m(x)^p\Big)^{1/p}. \tag{11}$$

If $\mathcal{E}$ is exponentially localized, (a) — (c) hold simultaneously for the larger class of all subexponential weights.

3. FRAME ANALYSIS OF THE QUASI-BANACH SPACES $\mathcal{H}^p_m, 0 < p < 1$

In this section we extend Theorem 2.2 to the quasi-Banach $\mathcal{H}^p_m, p < 1$. The extension of Theorem 2.2 to the case of quasi-Banach spaces is necessary for several concrete reasons: (a) In approximation theory the parameter p describes the "sparsity" of a representation, and there is no reason to limit the sparsity to $p \geq 1$, when higher sparsity (corresponding to smaller p) is of practical interest. (b) In the theory of Besov spaces as well as in other families of function spaces it is unnatural to restrict to the Banach space case. The complete results should be formulated for the full range of parameters.

To extend Theorem 2.2 to $p < 1$, we need to analyze the boundedness of the coefficient and reconstruction operators for the frame $\mathcal{E}$. Let $Cf = C_{\mathcal{E}}f = (\langle f, e_x\rangle)_{x\in\mathcal{X}}$ be the coefficient operator associated to $\mathcal{E}$, and $R\mathbf{c} = R_{\mathcal{E}}\mathbf{c} = \sum_{x\in\mathcal{X}} c_x e_x$ for $\mathbf{c} = (c_x)_{x\in\mathcal{X}}$ be the reconstruction operator. Then $S = S_{\mathcal{E}} = R_{\mathcal{E}}C_{\mathcal{E}}$. To prove a version of Theorem 2.2 for $\mathcal{H}^p_m, p < 1$, we have to show that both C and R are well-defined in the quasi-Banach space case. Although formally we have $R = C^*$, we have to prove the boundedness of R and C separately, because duality arguments do no longer work for $p < 1$.

The first lemma is an extension of [27, Lemma 2.1] to the case $0 < p < 1$. If $\mathcal{N} = \mathcal{X} = \mathbb{Z}^d$, the following lemma coincides with Young's Theorem $\ell^p_m * \ell^p_v \hookrightarrow \ell^p_m$ for $0 < p \le 1$.

Lemma 3.1. *Let $A = (A_{xn}), x \in \mathcal{X}, n \in \mathcal{N}$ be an $\mathcal{X}\times\mathcal{N}$-matrix with associated operator $(A\mathbf{c})(x) = \sum_{n\in\mathcal{N}} A_{xn}c_n$, and let $0 < p \le 1$.*

(a) If for some $\epsilon > 0$ $|A_{xn}| \le C(1 + |x - n|)^{-s-(d+\epsilon)/p}, \forall n \in \mathcal{N}, x \in \mathcal{X}$, then A is bounded from $\ell^p_m(\mathcal{N})$ to $\ell^p_m(\mathcal{X})$ for all s-moderate weights m.

(b) If $|A_{xn}| \le Ce^{-\alpha|x-n|}, x \in \mathcal{X}, n \in \mathcal{N}$, then A maps $\ell^p_m(\mathcal{N})$ to $\ell^p_m(\mathcal{X})$ for all subexponential weights m.

Proof. (a) Since $|\sum_{x\in\mathcal{X}} c_x|^p \le \sum_{x\in\mathcal{X}} |c_x|^p$ for $0 < p \le 1$, we can argue as in [27, Lemma 2.1].

$$\begin{aligned}
\|A\mathbf{c}\|^p_{\ell^p_m(\mathcal{X})} &= \sum_{x\in\mathcal{X}} \Big|\sum_{n\in\mathcal{N}} A_{xn}c_n\Big|^p m(x)^p \\
&\le C\sum_{x\in\mathcal{X}}\sum_{n\in\mathcal{N}} (1 + |x - n|)^{-sp-(d+\epsilon)}|c_n|^p\, m(x)^p \\
&\le C \sup_{n\in\mathcal{N}} \Big(\sum_{x\in\mathcal{X}} (1 + |x - n|)^{-(d+\epsilon)}\Big) \Big(\sup_{x\in\mathcal{X},n\in\mathcal{N}} (1 + |x - n|)^{-sp}m(n)^{-p}m(x)^p\Big) \qquad (12)\\
&\qquad \times \Big(\sum_{n\in\mathcal{N}} |c_n|^p m(n)^p\Big) \\
&= C' \,\|\mathbf{c}\|^p_{\ell^p_m(\mathcal{N})}\,.
\end{aligned}$$

Since $\mathcal{X}$ is relatively separated, the first supremum occurring in (12) is finite (see also [27, Lemma 2.1]), the second supremum is finite because of $m(x - n + n)^p \le C(1 + |x - n|)^{ps}m(n)^p$ for all $x, n \in \mathbb{R}^d$.

(b) is shown similarly and left to the reader. ■

We now show that the frame operator of localized frames is well behaved on the quasi-Banach spaces $\mathcal{H}^p_m, 0 < p < 1$. For polynomial localization the results hold only above a critical index p_0, whereas for exponential localization there is not restriction on p.

Proposition 3.2. *Assume that $\mathcal{E}$ is an r-localized frame for some $r > s + d$. Let p_0 be the critical index $p_0 = \frac{d}{r-s} < 1$.*

If $p_0 < p \le \infty$ and m is an s-moderate weight, then

(a) the coefficient operator $C_{\mathcal{E}}$ is bounded from $\mathcal{H}^p_m$ to $\ell^p_m(\mathcal{X})$,

(b) the synthesis operator $R_{\mathcal{E}}$ *extends to a bounded mapping from* $\ell^p_m(\mathcal{X})$ *to* $\mathcal{H}^p_m$*, and*

(c) the frame operator $S = S_{\mathcal{E}} = D_{\mathcal{E}} C_{\mathcal{E}}$ *maps* $\mathcal{H}^p_m$ *into* $\mathcal{H}^p_m$*, and the series converges unconditionally for* $p_0 < p < \infty$*.*

If $\mathcal{E}$ *is an exponentially localized frame , then these statements hold for all subexponential weights and all* $p, 0 < p \leq \infty$*.*

Proof. We prove these statements for polynomially localized frames and leave the simple modifications for exponentially localized frames to the reader. The case $p \geq 1$ is already contained in [27], so we may assume $p \leq 1$.

(a) Set $A_{xn} = \langle g_n, e_x \rangle$. If $f = \sum_{n \in \mathcal{N}} c_n g_n$, then

$$|(Cf)(x)| = |\langle f, e_x \rangle| = \Big| \sum_{n \in \mathcal{N}} c_n \langle g_n, e_x \rangle \Big| = (A\mathbf{c})(x) \,. \tag{13}$$

By hypothesis, $|A_{xn}| \leq C(1+|x-n|)^{-r}$, so Lemma 3.1(a) implies that A is bounded on $\ell^p_m(\mathcal{N})$, provided that $r > s + d/p$ or $p > \frac{d}{r-s} = p_0$.

Consequently,

$$\|C_{\mathcal{E}} f\|_{\ell^p_m(\mathcal{X})} = \|A\mathbf{c}\|_{\ell^p_m(\mathcal{N})} \leq C' \|\mathbf{c}\|_{\ell^p_m(\mathcal{N})} = C' \|f\|_{\mathcal{H}^p_m} \,.$$

(b) Set $b_{nx} = \langle e_x, \tilde{g_n} \rangle$. We need to show that for $\mathbf{c} \in \ell^p_m(\mathcal{X})$ the sequence with entries

$$(R\mathbf{c})(n) = \langle \sum_{x \in \mathcal{X}} c_x e_x, \tilde{g_n} \rangle = (B\mathbf{c})(n)$$

is in $\ell^p_m(\mathcal{N})$. As above this follows from the hypothesis and Lemma 3.1(a) (with $\mathcal{X}$ and $\mathcal{N}$ interchanged).

(c) follows by combining (a) and (b). As for the unconditional convergence of the series defining S, let $\epsilon > 0$ and choose $\mathcal{N}_0 = \mathcal{N}_0(\epsilon)$, such that $\|\langle f, e_x \rangle_{x \notin \mathcal{N}_0}\|_{\ell^p_m} \leq \epsilon$. Then for any finite set $\mathcal{N}_1 \supseteq \mathcal{N}_0$, assertions (a) and (b) imply that

$$\|Sf - \sum_{x \in \mathcal{N}_1} \langle f, e_x \rangle e_x\|_{\mathcal{H}^p_m} \quad \leq \quad \|R_{\mathcal{E}}\|_{op} \, \|\langle f, e_x \rangle_{x \notin \mathcal{N}_1}\|_{\ell^p_m} < \|R_{\mathcal{E}}\|_{op} \, \epsilon \,.$$

This means that $\sum_{x \in \mathcal{X}} \langle f, e_x \rangle e_x$ converges unconditionally in $\mathcal{H}^p_m$. ■

We can now formulate the main result about the frame characterization for the quasi-Banach spaces $\mathcal{H}^p_m, p < 1$.

Theorem 3.3. *Assume that* $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ *is an* r*-localized frame with respect to the Riesz basis* $\{g_n\}$ *and that* $r > s + d$*. Set* $p_0 = \frac{d}{r-s} < 1$*.*

(a) Then the frame operator S *is invertible simultaneously on all quasi-Banach spaces* $\mathcal{H}^p_m$*, where* $p_0 < p \leq \infty$ *and* m *is an* s*-moderate weight.*

(b) The frame expansion

$$f = \sum_{x \in \mathcal{X}} \langle f, e_x \rangle \tilde{e_x} = \sum_{x \in \mathcal{X}} \langle f, \tilde{e_x} \rangle e_x \tag{14}$$

converges unconditionally in $\mathcal{H}^p_m$ *for* $p_0 < p < \infty$ *(and weak* unconditionally in* $\mathcal{H}^\infty_m$*).*

(c) We have the norm equivalence

$$\|f\|_{\mathcal{H}^p_m} \asymp \Big(\sum_{x\in\mathcal{X}} |\langle f, e_x\rangle|^p m(x)^p\Big)^{1/p} \asymp \Big(\sum_{x\in\mathcal{X}} |\langle f, \widetilde{e_x}\rangle|^p m(x)^p\Big)^{1/p} \tag{15}$$

If $\mathcal{E}$ is an exponentially localized frame , then (a) — (c) hold simultaneously for all subexponential weights and all $p, 0 < p \le \infty$.

Proof. Since $S = \Gamma^{-1}T\Gamma$ by (5) and since by definition Γ is an isometric isomorphism between $\mathcal{H}^p_m$ and $\ell^p_m(\mathcal{N})$, S is invertible on $\mathcal{H}^p_m$ if and only if T is invertible on $\ell^p_m(\mathcal{N})$. We have seen in the proof of Theorem 2.1 that $T^{-1} \in \mathcal{A}_r$ whenever $\mathcal{E}$ is r-localized, therefore T^{-1} is bounded simultaneously on all $\ell^p_m(\mathcal{N})$ for $p > p_0$ and s-moderate weights by Lemma 3.1. This proves (a).

(b) follows from Proposition 3.2, since the identity on $\mathcal{H}^p_m$ factors as $\mathrm{I} = R_{\widetilde{\mathcal{E}}}C_{\mathcal{E}}$ and both $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are r-localized and $r > s + d/p$.

(c) This factorization also implies the norm equivalence: Since $f = R_{\widetilde{\mathcal{E}}}C_{\mathcal{E}}f$, we have

$$\|f\|_{\mathcal{H}^p_m} \le \|R_{\widetilde{\mathcal{E}}}\|_{op}\|C_{\mathcal{E}}f\|_{\ell^p_m(\mathcal{X})} \le \|R_{\widetilde{\mathcal{E}}}\|_{op}\|C_{\mathcal{E}}\|_{op}\|f\|_{\mathcal{H}^p_m} .$$

The second norm equivalence is shown by using the factorization $\mathrm{I} = R_{\mathcal{E}}C_{\widetilde{\mathcal{E}}}$. ■

REMARK: Note that for the treatment of the Banach spaces $\mathcal{H}^p_m$ with s-moderate weights we need r-localized frames with $r > s + d$. Consequently, we have always $p_0 = \frac{d}{r-s} < 1$.

4. Non-Linear Approximation with Localized Frames

The main question in non-linear approximation theory is how well elements in a Banach space can be approximated by a given dictionary. Good approximation properties usually yield sparse representations with respect to the given dictionary. See [13, 35] for an extended discussion of the state of art of non-linear approximation.

In this brief section we investigate the approximation in the Banach spaces $\mathcal{H}^p_m$ when the dictionary is a localized frame $\mathcal{E}$. More precisely, we consider so-called N-term approximation in $\mathcal{H}^p_m$ with respect to $\mathcal{E}$.

Definition 3. For a frame $\mathcal{E}$ of $\mathcal{H}$ we let

$$\Sigma_N = \Big\{p = \sum_{x\in F} c_x e_x \ : F \subseteq \mathcal{X}, \mathrm{card}\ F \le N\Big\}$$

denote the set of all linear combinations consisting of at most N terms. The N-term approximation error in $\mathcal{H}$ is defined by

$$\sigma_N(f) = \inf_{p\in\Sigma_N} \|f - p\|_{\mathcal{H}} .$$

The main tool for investigation N-term approximation is the following lemma of Stechkin [33] and deVore–Temlyakov [14].

Lemma 4.1. *Assume that* $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$, $0 < p < 2$, *and set* $\alpha = \frac{1}{p} - \frac{1}{2}$ *and* $\sigma_N(\mathbf{a}) = (\sum_{j=N+1}^{\infty} |a_j|^2)^{1/2}$. *Then there is a constant* $C = C(p) > 0$, *such that*

$$\frac{1}{C} \|\mathbf{a}\|_p \leq \Big(\sum_{N=1}^{\infty} \big(N^\alpha \sigma_{N-1}(\mathbf{a})\big)^p \frac{1}{N} \Big)^{1/p} \leq C \|\mathbf{a}\|_p .$$

The following theorem shows that localized frames possess the same approximation power as the underlying Riesz basis.

Theorem 4.2. *Assume that* $\mathcal{E}$ *is* s*-localized with respect to the Riesz basis* $\{g_n : n \in \mathcal{N}\}$ *and that* $s > d$. *Set* $\alpha = \frac{1}{p} - \frac{1}{2} > 0$ *for* $p < 2$ *and* $p_0 = \frac{d}{s} < 1$.

If $p_0 < p < 2$ *and* $f \in \mathcal{H}^p$, *then*

$$\Big(\sum_{N=1}^{\infty} (N^\alpha \sigma_{N-1}(f))^p \frac{1}{N} \Big)^{1/p} \leq C \|f\|_{\mathcal{H}^p} . \tag{16}$$

In particular, if $f \in \mathcal{H}^p$ *and* $\frac{d}{s} < p < 2$, *then* $\sigma_N(f) = \mathcal{O}(N^{-\alpha})$.

If $\mathcal{E}$ *is exponentially localized, then* (16) *holds for all* $p, 0 < p < 2$.

Proof. By Theorems 2.2 and 3.3 every $f \in \mathcal{H}^p$, has the frame expansion $f = \sum_{x \in \mathcal{X}} \langle f, \widetilde{e_x} \rangle e_x$ with coefficients $\mathbf{c} = (c_x) := \big(\langle f, \widetilde{e_x} \rangle \big)_{x \in \mathcal{X}} \in \ell^p(\mathcal{X})$ and $\|\mathbf{c}\|_p \asymp \|f\|_{\mathcal{H}^p}$. Let $|c_{x_1}| \geq |c_{x_2}| \geq |c_{x_3}| \geq \cdots \geq 0$ be a non-increasing rearrangement of these coefficients and set $p_N = \sum_{j=1}^{N} c_{x_j} e_{x_j} \in \Sigma_N$. Then

$$\begin{aligned} \sigma_N(f) &\leq \|f - p_N\|_{\mathcal{H}} \\ &= \| \sum_{j=N+1}^{\infty} c_{x_j} e_{x_j} \|_{\mathcal{H}} \\ &\leq C \Big(\sum_{j=N+1}^{\infty} |c_{x_j}|^2 \Big)^{1/2} = \sigma_N(\mathbf{c}) \end{aligned}$$

Since the characterization (11) of $\mathcal{H}^p$ (with trivial weight) holds exactly for $\frac{d}{s} < p < 2$, we find that

$$\begin{aligned} \Big(\sum_{N=1}^{\infty} (N^\alpha \sigma_{N-1}(f))^p \frac{1}{N} \Big)^{1/p} &\leq \Big(\sum_{N=1}^{\infty} (N^\alpha \sigma_{N-1}(\mathbf{c}))^p \frac{1}{N} \Big)^{1/p} \\ &\leq C \|\mathbf{c}\|_p \\ &\leq C' \|f\|_{\mathcal{H}^p} \end{aligned}$$

after invoking Lemma 4.1. For exponentially localized frames, (15) and therefore the statement hold for $0 < p < 2$. ∎

REMARKS: 1. It is an open problem whether the converse of (16) holds. If we use the Riesz basis $\{g_n\}$ as our dictionary, then Lemma 4.1 implies immediately that $\Big(\sum_{N=1}^{\infty} (N^\alpha \sigma_N(f))^p \frac{1}{N} \Big)^{1/p} \asymp \|f\|_{\mathcal{H}^p}$, see [14]. To prove this equivalence for localized frames, we would need a Bernstein-type inequality for finite linear combinations of frame elements. An additional obstacle is that the largest frame coefficients do not necessarily provide the optimal N-term approximation. However, there is some hope that the converse inequality in (16) is true and that the spaces $\mathcal{H}^p$ can be characterized by their approximation properties with respect to localized frames, because every localized frame is a finite union of Riesz sequences [26] and since a Bernstein-type inequality holds for Riesz sequence.

For another interpretation of Theorem 4.2 and a much more general theory of non-linear approximation the reader should consult the work of Gribonval and Nielsen [20, 21].

3. Note that for polynomial localization, there is a saturation effect in the approximation power, given by $p_0 = \frac{d}{s} < 1$, whereas for exponential localization this does not happen.

4. At this time it is not clear how the redundancy affects the properties of N-term approximation. This seems an interesting question in its own right.

5. Convergence of the Iterative Frame Algorithm

For a frame with frame bounds $A, B > 0$, the inverse frame operator can be calculated (at least in principle) by the Neumann series

$$S^{-1} = \alpha \sum_{k=0}^{\infty} (\mathrm{I} - \alpha S)^k \tag{17}$$

for arbitrary relaxation parameter $\alpha < 2/B$ [15]. This geometric series converges in the operator norm on $\mathcal{B}(\mathcal{H})$. It is useful to recast the frame reconstruction as an iterative algorithm as follows:

$$\begin{aligned} f_0 &= Sf = \sum_{x \in \mathcal{X}} \langle f, e_x \rangle e_x \\ f_{n+1} &= f_n + \alpha S(f - f_n) \qquad n \geq 0 \,. \end{aligned} \tag{18}$$

Then $f = \lim_{n\to\infty} f_n$ is the reconstruction of f from the frame coefficients $\langle f, e_x \rangle$.

In practice much more efficient algorithms are used, for instance, conjugate gradient acceleration and/or exploitation of the additional structure of a frame [24, 34], but (18) is still the starting point for the analysis of iterative frame algorithms.

Therefore an important question is whether this algorithm also converges in other norms. Specifically, if $\mathcal{E}$ is localized, does the series (17) also converge in the norm of the associated Banach spaces $\mathcal{H}^p_m$? In the light of Theorem 4.2, this amounts to asking whether the iterative frame algorithm preserves the sparsity of a representation.

Using the methods developed for localized frames so far, we can now answer this question affirmatively.

Theorem 5.1. *If $\mathcal{E}$ is r-localized with respect to a Riesz basis for $r > s + d$, then the Neumann series* (17) *converges in the operator norm of $\mathcal{H}^p_m$ and the iterative algorithm* (18) *converges in $\mathcal{H}^p_m$ for $1 \leq p \leq \infty$ and all s-moderate weights m. (If*

$\mathcal{E}$ is exponentially localized, then the convergence is in $\mathcal{H}^p_m$ for all subexponential weights).

Proof. As for Theorem 2.1, the core of this proof is a Banach algebra argument.

Step 1. We take up the notation of Section 2. Since Γ defined by $(\Gamma f)(n) = \langle f, \widetilde{g_n}\rangle$ is an isometric isomorphism between $\mathcal{H}^p_m$ and $\ell^p_m(\mathcal{N})$, and since $S = \Gamma^{-1}T\Gamma$ from (5), the convergence of the series (17) in the operator norm on $\mathcal{H}^p_m$ is equivalent to the convergence of the series

$$T^{-1} = \alpha \sum_{k=0}^{\infty} (\mathrm{I} - \alpha T)^k \tag{19}$$

in the operator norm on $\ell^p_m(\mathcal{N})$.

Step 2. Recall that $\mathcal{A}_r$ is the matrix algebra defined by the decay condition (4). Let $\sigma_{\ell^p_m}(A)$ be the spectrum of the operator A acting on $\ell^p_m(\mathcal{N})$ and $r_{\ell^p_m}(A) = \max\{|\lambda| : \lambda \in \sigma_{\ell^p_m}(A)\}$ be the corresponding spectral radius. If $A \in \mathcal{A}_r$ and $\lambda \in \mathbb{C}$, then $A - \lambda\mathrm{I} \in \mathcal{A}_r$ is bounded simultaneously on all ℓ^p_m for $1 \le p \le \infty$ and all s-moderate weights m by Lemma 3.1. Now assume that $\lambda \notin \sigma_{\ell^2}(A)$, i.e., $A - \lambda\mathrm{I}$ is invertible on $\ell^2(\mathcal{N})$. Then Jaffard's Lemma [30] implies that $(A - \lambda\mathrm{I})^{-1} \in \mathcal{A}_r$ and thus $(A - \lambda\mathrm{I})^{-1}$ is also bounded on all ℓ^p_m, i.e., $\lambda \notin \sigma_{\ell^p_m}(A)$. In conclusion, Jaffard's Lemma implies that $\sigma_{\ell^p_m}(A) \subseteq \sigma_{\ell^2}(A)$, and in particular

$$r_{\ell^p_m}(A) \le r_{\ell^2}(A)\,. \tag{20}$$

Step 3. Now we apply (20) to the operator $\mathrm{I} - \alpha T$ occuring in (19). The geometric series (19) converges on ℓ^2 because $\|\mathrm{I} - \alpha T\|_{op} = r_{\ell^2}(\mathrm{I} - \alpha T) < 1$ for $\alpha < 2/B$. By (20) we also have $r_{\ell^p_m}(\mathrm{I} - \alpha T) < 1$. This inequality for the spectral radius suffices to guarantee the convergence of (19) in the operator norm on ℓ^p_m. ∎

REMARK: In our opinion, this statement is of immense practical importance, because the convergence of the frame algorithm (and its accelerations) in different norms expresses a strong form of numerical stability of the frame algorithm. Theorem 5.1 says that for data $(\langle f, e_x\rangle)$ in a subspace of $\ell^2(\mathcal{X})$ *the frame algorithm converges automatically in the correct norm.*

6. LOCALIZATION WITH RESPECT TO RIESZ BASES WITH MULTIPLE GENERATORS

In applications of frame theory to shift-invariant spaces with multiple generators and to wavelet theory in higher dimensions, we encounter Riesz bases with structured index sets. In this section we deal with some technical modifications of the general theory and, as an example, we treat the sampling problem in shift-invariant spaces with multiple generators. We will keep the treatment elementary, but it is certainly possible to generalize these results further by using the concept of block bases and a vector-valued version of localization.

We assume that the Riesz basis for $\mathcal{H}$ is of the form

$$\mathcal{G} := \{g_{ln} : n \in \mathcal{N}, l = 1, \dots L\}$$

with dual basis $\{\widetilde{g_{ln}}, n \in \mathcal{N}, l = 1, \dots, L\}$. Thus the index set is $\mathcal{N} \times F$ where $\mathcal{N}$ is a separated set in $\mathbb{R}^d$ (usually a subset of $\mathbb{Z}^d$) and $F = \{1, 2, \dots, L\}$ is a finite set. Then we call a frame $\mathcal{E} = \{e_x : x \in \mathcal{X}\}$ for $\mathcal{H}$ s-localized with respect to the Riesz basis $\mathcal{G}$, if

$$\max_{l=1,\dots,L} |\langle e_x, g_{ln}\rangle| \le C(1 + |x - n|)^{-s} \qquad \forall x \in \mathcal{X}, n \in \mathcal{N}\,, \tag{21}$$

$$\max_{l=1,\dots,L} |\langle e_x, \widetilde{g_{ln}}\rangle| \le C(1 + |x - n|)^{-s} \qquad \forall x \in \mathcal{X}, n \in \mathcal{N}\,, \tag{22}$$

Similarly we understand exponential localization.

The main Theorem 2.1 and its consequences do not apply directly to this variation of localization. To study localized frames with respect to a multiply generated Riesz basis, we use a trick and some re-indexing and reduce this situation to Theorem 2.1.

Let $u \in \mathbb{R}^d$, $|u| = 1$, $\epsilon > 0$. We define the map $j : \mathcal{N} \times F \to \mathbb{R}^d$ by

$$j(l, n) = n + \epsilon l u \qquad \forall n \in \mathcal{N}, l = 1, \dots, L\,.$$

Since $\mathcal{N}$ is separated, we may choose $\epsilon > 0$ small enough, such that j is one-to-one and the new index set $\widetilde{\mathcal{N}} := j(\mathcal{N} \times F) \subseteq \mathbb{R}^d$ is separated. If $m = j(l, n) \in \widetilde{\mathcal{N}}$, we write $h_m = h_{j(l,n)} := g_{ln}$. Clearly, the set $\{h_m : m \in \widetilde{\mathcal{N}}\}$ is a Riesz basis for $\mathcal{H}$, in fact, it is the same Riesz basis with relabeled index set. We check the localization with respect to $\{h_m\}$:

$$\begin{aligned}
|\langle e_x, h_m\rangle| &= |\langle e_x, g_{ln}\rangle| \\
&\le C\left(1 + |x - n|\right)^{-s} \\
&\le C\Big(\max_{l=1,\dots,L}\left(1 + |x - \epsilon l u|\right)^{s}\Big)\left(1 + |x - m|\right)^{-s} \\
&\le C'\left(1 + |x - m|\right)^{-s}.
\end{aligned}$$

The same estimate works for the dual Riesz basis, thus the frame $\mathcal{E}$ is localized with respect to the Riesz basis $\{h_m : m \in \widetilde{\mathcal{N}}\}$ in the sense of Definition 1. Conversely, localization with respect to the re-indexed Riesz basis $\{h_m\}$ implies localization with respect to the multiply generated Riesz basis $\{g_{ln}\}$.

Thus we can now reformulate the main Theorem 2.1 as follows.

Theorem 6.1. *If $\mathcal{E}$ is s-localized with respect to the Riesz basis $\{g_{ln} : n \in \mathcal{N}, l = 1, \dots L\}$ and if $s > d$, then $\widetilde{\mathcal{E}}$ is also s-localized.*

Likewise, if $\mathcal{E}$ is exponentially localized, then $\widetilde{\mathcal{E}}$ is also exponentially localized (with a possibly different exponent in (3)*).*

6.1. **Localization and Sampling theory.**

Let $\Phi = \{\phi_j : j = 1, \dots, L\} \subseteq L^2(\mathbb{R}^d)$ be a set of “generators” satisfying the following properties:

(i) The integer translates $\{\phi_l(\cdot - k) : k \in \mathbb{Z}^d, l = 1, \dots, L\}$ form a Riesz basis for the generated subspace in $L^2(\mathbb{R}^d)$. We also say that “Φ is a stable generator”.

(ii) Each ϕ_l is continuous.

(iii) Each ϕ_l satisfies the decay condition

$$|\phi_l(x)| \le C(1 + |x|)^{-r} \qquad \forall x \in \mathbb{R}^d\,, \tag{23}$$

for some $r > d$.

Given $0 < p \leq \infty$ and an s-moderate weight function m, we define the *multiply generated shift-invariant space* $V_m^p(\Phi)$ as

$$V_m^p(\Phi) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \ f = \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}^d} c_{ln} \phi_l(\cdot - n), \, \mathbf{c} \in (\ell_m^p(\mathbb{Z}^d))^L\} \, . \tag{24}$$

In other words, $V_m^p(\Phi)$ is the (quasi)-Banach space $\mathcal{H}_m^p$ associated to the multiply generated Riesz basis. Moreover, under the conditions stated, $V_m^p(\varphi)$ is a closed subspace of $L_m^p(\mathbb{R}^d)$ endowed with the equivalent norms

$$\|f\|_{L_m^p} \asymp \|\mathbf{c}\|_{\ell_m^p} \, . \tag{25}$$

For the general theory of shift-invariant spaces and sampling theory we refer to [1, 11, 27] and the references therein.

Since $V^2(\Phi)$ is shift-invariant, the dual basis is again of the form $\tilde{\Phi}_l(\cdot - n), l = 1, \ldots, L, n \in \mathbb{Z}^d$ for some $\tilde{\phi}_l \in V^2(\Phi)$. The following lemma is implicit in the literature on shift-invariant spaces, but does not seem to be sufficiently known. Here we give a new proof that is based on the theory of localized frames.

Lemma 6.2. *Under the hypotheses (i) — (iii) on* Φ*, the dual generators also satisfy the decay conditions*

$$|\widetilde{\phi_l}(x)| \leq C(1 + |x|)^{-r} \qquad \forall x \in \mathbb{R}^d, l = 1, \ldots, L \, . \tag{26}$$

Proof. Since a Riesz basis is also a frame, we can check the localization of the frame (Riesz basis) $\{\phi_l(\cdot - n), n \in \mathbb{Z}^d, l = 1, \ldots, L\}$ with respect to itself and then apply Theorem 6.1. In the following argument we use the easy estimate

$$\int_{\mathbb{R}^d} (1 + |t|)^{-r} (1 + |x - t|)^{-r} \, dt \leq C(1 + |x|)^{-r} \qquad \forall x \in \mathbb{R}^d \, , \tag{27}$$

which holds whenever $r > d$ (see for instance [25, Lemma 11.1.1]).

Step 1. We show that $\mathcal{E} = \{\phi_l(\cdot - n), n \in \mathbb{Z}^d, l = 1, \ldots, L\}$ is r-localized with respect to itself. We have

$$\langle \phi_l(\cdot - n), \widetilde{\phi_{l'}}(\cdot - m) \rangle = \delta_{l,l'} \delta_{mn}$$

by definition, and

$$|\langle \phi_l(\cdot - n), \phi_{l'}(\cdot - m) \rangle| \leq C \int_{\mathbb{R}^d} (1 + |x - n|)^{-r} (1 + |x - m|)^{-r} dt \leq C'(1 + |m - n|)^{-r}$$

by (23) and (27). So $\mathcal{E}$ is r-localized with respect to the Riesz basis $\mathcal{E}$.

Step 2. Theorem 6.1 implies that the dual frame $\widetilde{\mathcal{E}} = \{\widetilde{\phi_l}(\cdot - n)\}$, which in this case coincides with the dual basis, is also r-localized with respect to $\mathcal{E}$, in particular we have

$$|\langle \widetilde{\phi_l}, \widetilde{\phi_{l'}}(\cdot - n) \rangle| \leq C(1 + |n|)^{-r}$$

for $l, l' = 1, \ldots, L$, and $n \in \mathbb{Z}^d$.

Step 3. Expanding $\widetilde{\phi_l}$ with respect to the basis $\{\phi_l(\cdot - n)\}$, we obtain the desired decay estimate

$$\begin{aligned} |\widetilde{\phi_l}(x)| &= |\sum_{l'=1}^{L}\sum_{n\in\mathbb{Z}^d}\langle\widetilde{\phi_l},\widetilde{\phi_{l'}}(\cdot - n)\rangle\phi_{l'}(x-n)| \\ &\leq C\sum_{m\in\mathbb{Z}^d}(1+|n|)^{-r}(1+|x-n|)^{-r} \\ &\leq C'(1+|x|)^{-r} \qquad (28) \end{aligned}$$

for $l = 1,\dots,L$, where we have used the discrete version of (27) and (23) once more. ■

The main localization theorem for sampling in multiply generated shift-invariant spaces now goes as follows.

Theorem 6.3. *Assume that the generator Φ satisfies the assumptions (i) — (iii) for $r > s+d$. Let m be an s-moderate weight function and p_0 be the critical index $p_0 = \frac{d}{r-s}$.*

Assume that a relatively separated set $\mathcal{X} \subseteq \mathbb{R}^d$ satisfies the sampling inequality

$$A\|f\|_2 \leq \sum_{x\in\mathcal{X}}|f(x)|^2 \leq B\|f\|_2 \qquad \forall f\in V^2(\Phi) \tag{29}$$

for some constants $A, B > 0$. Then there exist dual functions $\widetilde{K}_x$ satisfying the following properties:

(a) Localization: *The $\widetilde{K}_x$ satisfy the estimates*

$$|\widetilde{K}_x(t)| \leq C(1+|t-x|)^{-r} \qquad \text{for all } x\in\mathcal{X}, t\in\mathbb{R}^d\,, \tag{30}$$

with a constant C independent of x and t.

(b) Local Reconstruction: *Every $f \in \mathcal{H}^p_m$ can be reconstructed by*

$$f = \sum_{x\in\mathcal{X}} f(x)\widetilde{K}_x \tag{31}$$

with unconditional convergence in $V^p_m(\Phi)$ for $p_0 = \frac{d}{r-s} < p < \infty$.

(c) Norm Equivalence:

$$A\|f\|_{L^p_m} \leq \Big(\sum_{x\in\mathcal{X}}|f(x)|^p m(x)^p\Big)^{1/p} \leq B\|f\|_{L^p_m} \qquad \forall f\in V^p_m(\Phi), p_0 < p \leq \infty\,. \tag{32}$$

REMARK: 1. For exponential localization $|\phi_j(x)| \leq Ce^{-\alpha|x|}$ we obtain $|\widetilde{K}_x(t)| \leq Ce^{-\beta|t-x|}$ for some $\beta\in(0,\alpha)$. The conclusions of Theorem 6.3 then hold for the full range $0 < p \leq \infty$ and all subexponential weights.

2. Since the precise nature of the functionals $f \to f(x)$ does not matter, the theorem also holds for sampling from local averages as in [27].

Proof. Under the assumptions stated on Φ, the Hilbert space $V^2(\Phi)$ is a reproducing kernel Hilbert space, and there exist functions $K_x \in V^2(\Phi)$ such that $f(x) =$

$\langle f, K_x\rangle$ [1]. The sampling inequality (29) simply says that $\{K_x : x \in \mathcal{X}\}$ is a frame for $V^2(\Phi)$. We check the localization properties of this frame with respect to the given Riesz basis:

$$|\langle K_x, \phi_l(\cdot - n)\rangle| = |\phi_l(x-n)| \le C(1+|x-n|)^{-r}$$

for $x \in \mathcal{X}, n \in \mathbb{Z}^d, l = 1, \dots, L$. Likewise, with the help of Lemma 6.2 we obtain that

$$|\langle K_x, \widetilde{\phi}_l(\cdot - n)\rangle| = |\widetilde{\phi}_l(x-n)| \le C(1+|x-n|)^{-r}\,.$$

This means that K_x is r-localized with respect to the multiply generated Riesz basis $\{\phi_l(\cdot - n) : n \in \mathbb{Z}^d, l = 1, \dots, L\}$.

We can now apply Theorem 2.1 and deduce that the dual frame $\widetilde{K}_x$ satisfies the estimates

$$|\langle \widetilde{K}_x, \phi_l(\cdot - n)\rangle| \le C'(1+|x-n|)^{-r}$$

and the same for the dual generators $\tilde{\phi}_l$. Since $\widetilde{K}_x = \sum_{l=1}^L \sum_{k\in\mathbb{Z}^d} \langle \widetilde{K}_x, \phi_l(\cdot - n)\rangle \tilde{\phi}_l(\cdot - n)$, the claimed decay (30) follows as in (28). The remaining assertions are now just a reformulation of the main Theorems 2.2 and 3.3. ■

7. Localized Wavelet Frames and Besov Spaces

Finally we deal with the characterization of Besov spaces by means of wavelet frames. While theorems of this type actually precede wavelet theory [17], a suitably formulated version of these results can now be obtained as a simple example of the general theory of localized frames.

Wavelet frames are frames for $L^2(\mathbb{R}^d)$ with a translation-dilation structure, i.e., $\mathcal{E} = \{\psi_{jkl}(x) = 2^{jd/2}\psi_l(2^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d, l = 1, \dots L\}$. In this example the obvious type of basis to compare with is an orthonormal wavelet basis, which is again of the form $\{\Psi_{jk\epsilon}(x) = 2^{jd/2}\Psi_\epsilon(2^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d, \epsilon = 1, \dots 2^d - 1\}$. For convenience we choose a Meyer type basis with wavelets $\Psi_\epsilon \in C^\infty(\mathbb{R}^d)$ with supp $\widehat{\Psi_\epsilon}$ being compact [31]. It is well-known that such wavelet bases are unconditional bases for all Besov-Triebel-Lizorkin spaces. More precisely, let $\ell^{p,q}_\alpha$ be the mixed norm space defined by the norm $\|\mathbf{c}\|_{\ell^{p,q}_\alpha} = \Big(\sum_{j\in\mathbb{Z}} \big(\sum_{k\in\mathbb{Z}^d}\sum_{\epsilon=1}^{2^d-1} |c_{jk\epsilon}|^p\big)^{q/p} 2^{j\alpha q}\Big)^{1/q}$. Let

$$\mathcal{H}^{p,q}_\alpha = \{f : f = \sum_{(j,k)\in\mathbb{Z}^{d+1}} \sum_{\epsilon=1}^{2^d-1} c_{jk\epsilon}\Psi_{jk\epsilon} : \mathbf{c} \in \ell^{p,q}_\alpha\}$$

be the Banach space associated to the Meyer-type wavelet basis with norm $\|f\|_{\mathcal{H}^{p,q}_\alpha} = \|\mathbf{c}\|_{\ell^{p,q}_\alpha}$. By the results in [22, 19, 31] we have the following identification with the homogeneous Besov spaces:

$$\mathcal{H}^{p,q}_\alpha = \dot{B}^{p,q}_{\alpha + \frac{d}{p} - \frac{d}{2}} \tag{33}$$

Theorem 7.1. *Assume that* $\{\psi_{jkl}(x) = 2^{jd/2}\psi_l(2^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d, l = 1, \dots L\}$ *is a frame for* $L^2(\mathbb{R}^d)$ *that satisfies the localization condition*

$$|\langle \psi_{jkl}, \Psi_{j'k'\epsilon}\rangle| \le C\big(1+|j-j'|+|k-k'|\big)^{-s} \qquad \forall j,j' \in \mathbb{Z}, k,k' \in \mathbb{Z}^d, l, \epsilon\,, \tag{34}$$

for some $s > d+1$. *Then the dual frame* $\widetilde{\psi_{jkl}}$ *satisfies the same localization conditions*

$$|\langle \widetilde{\psi_{jkl}, \Psi_{j'k'\epsilon}}\rangle| \leq C'\big(1+|j-j'|+|k-k'|\big)^{-s} \qquad \forall j,j' \in \mathbb{Z}, k,k' \in \mathbb{Z}^d, l, \epsilon\,. \tag{35}$$

Then a distribution f *is in* $\dot{B}^{p,q}_{\frac{d}{p}-\frac{d}{2}}$ *for* $p_0 = \frac{d}{s} < p, q \leq \infty$ *if and only if*

$$\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}^d, l\in F}|\langle f, \psi_{jkl}\rangle|^p\Big)^{q/p}\Big)^{1/q} < \infty\,.$$

The latter expression is an equivalent norm on $\dot{B}^{p,q}_{\frac{d}{p}-\frac{d}{2}}(\mathbb{R}^d)$.

Furthermore, if $f \in \dot{B}^{p,p}_{\frac{d}{p}-\frac{d}{2}}$ *for* $p_0 = \frac{d}{s} < p < 2$ *and* $\alpha = 1/p - 1/2$, *then*

$$\Big(\sum_{N=1}^{\infty}(N^{\alpha}\sigma_{N-1}(f))^p\,\frac{1}{N}\Big)^{1/p} \leq C\|f\|_{B^{p,p}_{\frac{d}{p}-\frac{d}{2}}}\,,$$

where $\sigma_N(f)$ *is the* N*-term approximation error of* f *with respect to the wavelet frame as in Definition 3.*

Proof. We choose the index sets to be $\mathcal{N} = \mathbb{Z}^{d+1} \times \{1,\dots,2^d-1\}$ and $\mathcal{X} = \mathbb{Z}^{d+1}$ (with multiplicity L). For $p = q$ the result is just an explicit formulation of Theorems 2.1, 2.2, 3.3, and 6.1 for wavelet frames and the associated Banach spaces $\mathcal{H}^p = \dot{B}^{p,p}_{\frac{d}{p}-\frac{d}{2}}$ (with weight $m \equiv 1$). The result on non-linear approximation follows from Theorem 4.2. The case $p \neq q$ requires a simple adaption of Proposition 3.2 to mixed norm spaces. Since the index sets are $\mathbb{Z}^{d+1}$, this amounts to replacing Lemma 3.1 by Young's Theorem $\ell^{p,q}_{\alpha} * \ell^{\min(1,p,q)}_{\alpha} \subseteq \ell^{p,q}_{\alpha}$ in the proof of Proposition 3.2. ■

REMARKS: 1. Since Theorem 7.1 resembles other results in the literature (compare [17, 18, 23]), let us point out some of the novelties: no assumption is made on the structure of the dual frame, in particular it is not assumed that the dual frame is again a wavelet frame. Likewise, the frame element ψ_{jkl} need not be of the form $\psi_l(2^j x - k)$, any frame satisfying (34) yields the same conclusions. This shows that the characterization of Besov spaces is more about the localization properties of frames than about the exact translation-dilation structure of the frames. Of course, this observation is consistent with the characterization of Besov-Triebel-Lizorkin spaces by molecules [18, 37].

2. Condition (34) implies in particular that the wavelet ψ satisfies $\sum_{j,k,\epsilon}|\langle\psi, \Psi_{jk\epsilon}\rangle| < \infty$. By (33) we have $\psi \in \dot{B}^{1,1}_{d/2}$ [31]. Thus the localization properties describes a subspace of $\psi \in \dot{B}^{1,1}_{d/2}$ as the class of admissible wavelets.

3. Condition (34) is satisfied when $\operatorname{supp}\widehat{\psi_l} \subseteq \{x \in \mathbb{R}^d : 0 < a \leq |x| \leq b\}$ and $|\psi_l(x)| = \mathcal{O}(|x|^{-s})$, because in this case $\langle\psi_{jkl}, \Psi_{j'k'\epsilon}\rangle = 0$ for $|j - j'|$ large enough. The condition is different from the conditions of Frazier-Jawerth in [17] and [18, Sec. 2 and 3], and neither implies nor is implied by their conditions. Other conditions can be found in [23].

4. Theorem 7.1 does not cover the entire range of Besov spaces. Using the lifting property of Besov spaces [36, p. 241] it is possible to show the following result. Let $\psi^\gamma, \gamma \in \mathbb{R}$, be defined by $\widehat{\psi^\gamma}(\omega) = |\omega|^\gamma \hat{\psi}(\omega)$ and fix $\beta > 0$. *Assume that* $\mathcal{E}_\gamma = \{\psi^\gamma_{jkl}(t) = 2^{jd/2}\psi^\gamma_l(2^j - k), j \in \mathbb{Z}, k \in \mathbb{Z}^d, l = 1, \dots L\}$ *is a frame for* $L^2(\mathbb{R}^d)$ *for* $\gamma \in \{0, -\beta, \beta\}$. *If* $\mathcal{E}$ *satisfies the localization condition*

(36)

$$|\langle \psi_{jkl}, \Psi_{j'k'\epsilon}\rangle| \le C\big(1 + |j - j'| + |k - k'|\big)^{-s} 2^{-\beta|j-j'|} \qquad \forall j, j' \in \mathbb{Z}, k, k' \in \mathbb{Z}^d, l, \epsilon\,,$$

for some $s > d + 1$, *then the dual frame* $\{\widetilde{\psi_{jkl}}\}$ *satisfies the same localization conditions. A distribution* f *is in* $\dot{B}^{p,q}_{\alpha + \frac{d}{p} - \frac{d}{2}}$ *for* $p_0 = \frac{d}{s} < p, q \le \infty$ *and* $|\alpha| \le \beta$ *if and only if*

$$\Big(\sum_{j\in\mathbb{Z}} \big(\sum_{k\in\mathbb{Z}^d, l\in F} |\langle f, \psi_{jkl}\rangle|^p\big)^{q/p} 2^{\alpha j q}\Big)^{1/q} < \infty\,.$$

The latter expression is an equivalent norm on $\dot{B}^{p,q}_{\alpha + \frac{d}{p} - \frac{d}{2}}$.

A detailed account of localized wavelet frames will be given elsewhere. The hypothesis that $\psi^{\pm\beta}$ generates a wavelet frame seems redundant, but so far we have not been able to remove it.

5. The "hard analysis" of Frazier-Jawerth [18] and Meyer [32, Ch. VIII.3,4] suggests an alternative approach to wavelet localization. Imitating the concept of almost diagonalization of operators with respect to wavelet bases, one may define the localization of a wavelet frame with respect to a Meyer wavelet basis by

$$|\langle \psi_{jkl}, \Psi_{j'k'\epsilon}\rangle| \le C 2^{-\beta|j-j'|}\, 2^{-s|j-j'|/2} \Big(1 + \frac{|2^{-j}k - 2^{-j'}k'|}{\max(2^{-j}, 2^{-j'})}\Big)^{-s}, \tag{37}$$

for all $j, j' \in \mathbb{Z}$, $k, k' \in \mathbb{Z}^d$, l, ϵ and for some $s > d$. As explained in [18, Rem. 3.2], this condition amounts to an exponential decay with respect to the hyperbolic metric on the upper half space $\mathbb{H} = \mathbb{R}^d \times \mathbb{R}^+$. More precisely, we identify $(j, k) \in \mathbb{Z}^{d+1}$ with the point $\lambda = (2^{-j}k, 2^{-j}) \in \mathbb{H}$ and endow $\mathbb{H}$ with a metric d that is right invariant under the $ax + b$-group, see [18, p. 53] for the exact formula. Then (37) can be written as

$$|\langle \psi_{jkl}, \Psi_{j'k'\epsilon}\rangle| \le C 2^{-\beta|j-j'|}\, e^{-s\, d(\lambda,\lambda')}\,,$$

i.e., exponential localization with respect to a different metric on the index set. While this approach lacks the simplicity of Theorem 7.1, it might lead to more profound results, in particular, we would expect that this type of localization also yields characterizations of the Triebel-Lizorkin spaces.

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