ELLIPTIC AND PARABOLIC SECOND-ORDER PDES WITH GROWING COEFFICIENTS

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Abstract. We consider a second-order parabolic equation in $\mathbb{R}^{d+1}$ with possibly unbounded lower order coefficients. All coefficients are assumed to be only measurable in the time variable and locally Hölder continuous in the space variables. We show that global Schauder estimates hold even in this case. The proof introduces a new localization procedure. Our results show that the constant appearing in the classical Schauder estimates is in fact independent of the L_∞-norms of the lower order coefficients. We also give a proof of uniqueness which is of independent interest even in the case of bounded coefficients.

1. Introduction

Let us consider the following second-order operator L,

$$Lu(t,x) = a^{ij}(t,x)u_{x^ix^j}(t,x) + b^i(t,x)u_{x^i}(t,x) - c(t,x)u(t,x), \quad (1.1)$$

acting on functions defined on $[T,\infty)\times\mathbb{R}^d$ if $T\in(-\infty,\infty)$ and on $\mathbb{R}^{d+1}$ if $T=-\infty$ (the summation convention is enforced throughout the article). We prove global Schauder estimates for solutions of the equation

$$u_t(t,x) + Lu(t,x) = f(t,x), \quad (t,x)\in(T,\infty)\times\mathbb{R}^d. \quad (1.2)$$

Roughly speaking, we will assume that a,b,c,f are measurable in t locally bounded with respect to (t,x), Hölder continuous in x in any ball of radius one with a constant independent of t and the position of the ball. Moreover, we assume that $c(t,x)$ is always greater than some constant $\delta>0$, f is pointwise "controlled" by c, and the matrix a is uniformly bounded and uniformly positive definite. The local Hölder continuity does not prevent b,c, and f from growing linearly as $|x|\to$

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∞. Thus, we do not assume that b, c, and f are globally bounded as in the classical setting (see [2], [10], [15], [16], [17]). Recently, the interest in elliptic and parabolic equations with unbounded coefficients in the whole space has increased (see, for instance, [3], [6], [7], [9], [18], [19], [20], [21] and the references therein). Such equations arise naturally also in stochastic control and filtering theory (see, for instance, [8] and [23]).

In Theorem 2.4, we obtain parabolic Schauder estimates of the type

$$\sup_{t\geq T}\|u(t,\cdot)\|_{2+\alpha}\leq N\sup_{t\geq T}\|f(t,\cdot)\|_{\alpha,\,loc},\tag{1.3}$$

by means of a new localization procedure and Lemma 3.10 saying that the constants in classical Schauder estimates for equations with coefficients depending only on t are independent of the magnitudes of b^i and c. In particular, from (1.3) we deduce new elliptic Schauder estimates when a^{ij}, b^i, c, and f do not depend on t. It is noteworthy that to prove the new elliptic Schauder estimates in $\mathbb{R}^d$ we need to use the corresponding result for *parabolic* equations in $\mathbb{R}^{d+1}$. Estimate (1.3) allows us to prove the solvability of (1.2) (Theorem 2.5), of the related Cauchy problem (Theorem 2.8), and of similar elliptic equations in the whole space (Theorem 2.6).

While dealing with equations with growing coefficients in the whole space it is quite natural to work in $C^{2+\alpha}$ spaces with weights which would not allow the derivatives of solutions to grow so that the terms $a^{ij}u_{x^ix^j}$, $b^iu_{x^i}$, and cu would remain in the usual C^α without weights (see, for instance, [4], [11], and the references therein).

Naturally, once we want to allow the coefficients to grow, the question arises as to what happens if the coefficients do not grow but f does. Such cases were investigated for instance in [17] (see also [13, Remark 2.2]) where solutions were looked for in $C^{2+\alpha}$ spaces with weights.

We discuss now some recent papers dealing with Schauder estimates for elliptic equations and for *autonomous* parabolic Cauchy problems involving unbounded coefficients.

In [4] elliptic and parabolic equations with unbounded coefficients are studied assuming a kind of "balance" between the first order term $b^iu_{x^i}$ and the potential term cu (a similar balance was also used in [1]). Schauder estimates in [4] follow by generation of analytic semigroup in Hölder spaces. Weighted Hölder spaces $C^{2+\alpha}(V)$ are introduced such that L becomes a bounded operator from $C^{2+\alpha}(V)$ onto the usual C^α with bounded inverse. Roughly speaking, V in [4] is a function comparable with c such that $|b|\leq V^{1/2}$, and $C^{2+\alpha}(V)$ is the space of functions such that Vu and u_{xx} belong to the usual C^α.

It was discovered in [7] that even if b grows linearly and $c = 0$ one can still have the solvability theory in the *usual* $C^{2+\alpha}$ spaces without weights (note that in this case, contrarily to [4], generation of analytic semigroup fails). In [7] it is assumed that a^{ij} are constant and $b(t,x) = Ax$, $x \in \mathbb{R}^d$, for some fixed $d \times d$ real matrix A.

The results of [7] are surprising for the following reason. For elliptic equations with bounded Hölder continuous coefficients we always have $LC^{2+\alpha} = C^\alpha$ and $C^{2+\alpha} = L^{-1}C^\alpha$. However, if we allow b to grow, then $L^{-1}C^\alpha \subset C^{2+\alpha}$ with proper inclusion and, even more than that, for different L the sets $L^{-1}C^\alpha$ may be different (we say more about this in Remark 2.7). This situation differs dramatically from what was common before. For instance, in [4] for all operators L, satisfying the conditions imposed there with the same V, the set $L^{-1}C^\alpha$ is always $C^{2+\alpha}(V)$.

After [7], several authors dealt with elliptic Schauder estimates and Schauder estimates for autonomous Cauchy problems with unbounded *smooth* coefficients, see, for instance, [3], [5], [6], [18], [19], [21]. These papers also contain regularity results not covered in the present article. However, our understanding is that their methods and results can not be used to derive Schauder estimates in our situation (even if we would consider our coefficients a, b, and c independent on t). For example, in [18] and [6] Schauder estimates are proved assuming that a^{ij} and b^i are smooth enough (since the methods used require to differentiate the equations three times). Note that mollifying the coefficients a^{ij} and then using , for instance, the results of [18] do not seem to allow one to get our elliptic Schauder estimates. In [19] Schauder estimates are proved assuming only that a^{ij} are bounded and have first order bounded derivatives but an additional compatibility condition between a^{ij} and b^i is imposed.

Classical (possibly non-autonomous) parabolic Schauder estimates when a^{ij}, b^i, and c are bounded and Hölder continuous in space and time are proved in [15]. Partial Schauder estimates like (1.3) in the case of bounded coefficients which are only measurable in time were discovered in [2]. Then in [10] it was shown that the second derivatives of the solution u in x are Hölder continuous with respect to the time variable. We will see below in Lemma 3.5 that this result is, actually, an embedding theorem and has little to do with parabolic equations. In [16] interior parabolic estimates and Schauder estimates in a bounded parabolic domain up to the boundary are proved. The parabolic Cauchy problem when the coefficients are bounded, discontinuous in time and Hölder continuous in space is investigated in [17].

It seems that our Theorem 2.5 on solvability of equation (1.2) has not been stated before even in the case of bounded coefficients. We also stress that uniqueness of solutions is based on an a priori estimate of independent interest (see Theorem 4.1) which seems to be new even when the coefficients are bounded. Our exposition is independent of the standard $C^{2+\alpha}$-theory of elliptic and parabolic equations and, actually, one can get basic Schauder estimates of this theory directly from our results (see Corollary 4.4).

Our main results are stated in Section 2 where we also prove two of them, namely Theorem 2.6 and 2.8. Theorems 2.4 and 2.5 are proved in Section 4 after we prepare necessary auxiliary results on equations with coefficients independent of x in Section 3.

2. Main results

Introduce

$$\mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R}, x = (x^1,...,x^d) \in \mathbb{R}^d\}.$$

Hypothesis 2.1. (i) The matrix $(a^{ij}(t,x))$ is symmetric and there exist constants $\delta, K \in (0,\infty)$ such that

$$KI \geq (a^{ij}(t,x)) \geq \delta I, \quad c(t,x) \geq \delta, \qquad (t,x) \in \mathbb{R}^{d+1}.$$

(ii) The functions $a^{ij}(t,x)$, $b^i(t,x)$, and $c(t,x)$ are measurable in $\mathbb{R}^{d+1}$; $b(t,0)$ and $c(t,0)$ are locally bounded in t, and, for some $\alpha \in (0,1)$,

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |b^i(t,x) - b^i(t,y)| + |c(t,x) - c(t,y)| \leq K|x-y|^\alpha$$

for all $t \in \mathbb{R}$, $i,j = 1,...,d$, and $x,y \in \mathbb{R}^d$ such that

$$|x-y| \leq 1. \tag{2.1}$$

Hypothesis 2.2. The function $f(t,x)$ is measurable in $\mathbb{R}^{d+1}$ and there exist constants F_0 and F_α such that we have

$$|f(t,x)| \leq F_0\, c(t,x) \tag{2.2}$$

$$|f(t,x) - f(t,y)| \leq F_\alpha |x-y|^\alpha,$$

whenever $t \in \mathbb{R}$, $x,y \in \mathbb{R}^d$ and (2.1) holds.

Our Theorem 2.8 about the Cauchy problem is also true when $c(t,x)$ is only nonnegative on $\mathbb{R}^{d+1}$. More precisely, such result holds if $c(t,x)$ is replaced by 1 in assumption (2.2) with the other hypotheses unchanged (see Remark 2.9).

As usual $C^\alpha = C^\alpha(\mathbb{R}^d)$ is the Banach space of real functions g on $\mathbb{R}^d$ with finite norm

$$\|g\|_\alpha = \|g\|_0 + [g]_\alpha,$$

where

$$\|g\|_0 = \sup_{x\in\mathbb{R}^d} |g(x)|, \quad [g]_\alpha = \sup_{x,y\in\mathbb{R}^d} \frac{|g(x)-g(y)|}{|x-y|^\alpha}, \quad \big(\tfrac{0}{0} := 0\big).$$

By $C^{2+\alpha}$ we denote the space of real functions u on $\mathbb{R}^d$ with finite norm

$$\|u\|_{2+\alpha} = \|u\|_0 + \|Du\|_0 + \|D^2u\|_0 + [u]_{2+\alpha},$$

where $Du = (D_1u, ..., D_du)$, $D^2u = (D_{ij}u; i,j = 1, ..., d)$,

$$D_iu = u_{x^i}, \quad D_{ij}u = D_iD_ju = u_{x^ix^j}, \quad [u]_{2+\alpha} = [D^2u]_\alpha.$$

For any $T \in (-\infty, \infty)$, we define

$$\mathbb{R}^{d+1}_T = \mathbb{R}^{d+1} \cap \{t \geq T\} = [T,\infty) \times \mathbb{R}^d,$$

and, if $T = -\infty$, we set $\mathbb{R}^{d+1}_T = \mathbb{R}^{d+1}$. The supremum norm of functions on $\mathbb{R}^d_T$ will be denoted by $\|\cdot\|_{0,T}$.

We will be working with the set $\mathcal{C}^{2+\alpha}(T)$ of functions $u(t,x)$ defined on $\mathbb{R}^{d+1}_T$ such that

(i) the function u is continuous in $\mathbb{R}^{d+1}_T$;

(ii) for each finite $t \geq T$, we have $u(t,\cdot) \in C^{2+\alpha}$ and $\|u(t,\cdot)\|_{2+\alpha}$ is bounded in t;

(iii) there is a measurable function $g(t,x)$ defined on $\mathbb{R}^{d+1}_T$ such that for any $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ the function $\zeta(t,x)g(t,x)$ is bounded and α-Hölder continuous in x with constant independent of t and for any $x \in \mathbb{R}^d$ and any finite s and t, such that $T \leq s < t$, we have

$$u(t,x) - u(s,x) = \int_s^t g(r,x)\,dr. \tag{2.3}$$

For such a function u we denote $u_t = g$. Obviously g is the generalized derivative of u with respect to t. By $\mathcal{C}^{2+\alpha}_0(T)$ we denote a subset of $\mathcal{C}^{2+\alpha}(T)$ consisting of functions vanishing for large $|x| + |t|$.

Recall that a continuous function u has a bounded generalized derivative with respect to a coordinate if and only if it is Lipschitz continuous with respect to this coordinate and if and only if u is absolutely continuous with respect to the coordinate and its classical derivative (existing almost everywhere) is bounded. Under any of the above conditions the classical derivative coincides with the generalized one and its essential supremum equals the Lipschitz constant.

Accordingly, solutions of equation (1.2) will be looked for in $\mathcal{C}^{2+\alpha}(T)$ and in this class (1.2) is equivalent to the fact that, for any $x \in \mathbb{R}^d$ and finite $t > s$ with $s \geq T$, we have

$$u(t,x) - u(s,x) = \int_s^t f(r,x)\,dr - \int_s^t Lu(r,x)\,dr. \tag{2.4}$$

Remark 2.3. Note that, according to Corollary 4.2, the unique solution u of (1.2) will have more regularity.

Here are our main results in which we always suppose that Hypotheses 2.1 and 2.2 are satisfied and $T \in [-\infty, \infty)$.

The first result bears on an a priori estimate.

Theorem 2.4. *Let $u \in \mathcal{C}^{2+\alpha}(T)$ satisfy* (1.2). *Then:*

(i) There is a constant $N = N(\delta, \alpha, d, K)$ such that, for all finite $t \geq T$, we have

$$\|u(t,\cdot)\|_{2+\alpha} \leq N(F_0 + F_\alpha). \tag{2.5}$$

(ii) If f and the coefficients of L are independent of t, then with the same N, for any $u \in C^{2+\alpha}$,

$$\|u\|_{2+\alpha} \leq N(F_0 + F_\alpha).$$

After we prove the solvability of model equations estimate (2.5) will allow us to prove the following existence theorem.

Theorem 2.5. *There exists a unique $u \in \mathcal{C}^{2+\alpha}(T)$ which satisfies* (1.2).

Theorem 2.5 allows us to treat elliptic equations.

Theorem 2.6. *Assume that the coefficients of L and f are independent of t. Then there exists a unique $u \in C^{2+\alpha}$ satisfying $Lu = f$.*

Proof. Let $u \in \mathcal{C}^{2+\alpha}(0)$ be a unique solution of the equation

$$u_t + Lu = f \tag{2.6}$$

in $\mathbb{R}^{d+1}_0$. Since the coefficients and f are independent of t, for any fixed $s \geq 0$, the function $u(s + \cdot, \cdot)$ also satisfies (2.6) in $\mathbb{R}^{d+1}_0$. Obviously $u(s + \cdot, \cdot) \in \mathcal{C}^{2+\alpha}(0)$. By uniqueness $u(s+t,x) = u(t,x)$ for any $t \geq 0$ and $x \in \mathbb{R}^d$. In particular, $u(s,x) = u(0,x)$, $s \geq 0$, and equation (2.6) becomes $Lu = f$. This gives the existence of solution. Uniqueness follows from assertion (ii) in Theorem 2.4. The theorem is proved.

Remark 2.7. As we have pointed out in the Introduction, a remarkable feature of this theorem is that the set $LC^{2+\alpha}$ is generally obviously wider than C^α unlike in the case that the coefficients of L are bounded when $LC^{2+\alpha} = C^\alpha$ always. Also notice that generally if c is bounded, the sets $L^{-1}C^\alpha$ are *different* for different L with growing coefficients because for a solution u of $Lu = f$ we have that $b^i D_i u$ is bounded. In addition, the boundedness of $b^i D_i u$ means that the projection of the gradient $Du(x)$ on $b(x)$ becomes smaller and smaller at points where $|b(x)|$ becomes large. What causes this remains a mystery. It is certainly not true that this happens only because $|Du(x)|$ is small where $|b(x)|$ is large, which is seen if we take $a^{ij} = \delta^{ij}$, $c = 1$, radially symmetric f, and *any* $b^i(x)$ such that $b^i(x)x^i \equiv 0$.

Our next main result concerns the Cauchy problem. Take and fix a finite $S > T$ and introduce the space $\mathcal{C}^{2+\alpha}(T, S)$ as the set of functions $u \in \mathcal{C}^{2+\alpha}(T)$ such that $u(t, x) = u(S, x)$ for $t \geq S$.

We will consider the equation

$$u_t + Lu = f \quad \text{in } (T, S) \times \mathbb{R}^d. \tag{2.7}$$

Clearly, we may assume that $f = 0$ in $(\mathbb{R} \setminus [T, S]) \times \mathbb{R}^d$. Therefore, concerning f it is enough to assume that

$$|f(t, x)| \leq F_0\, c(t, x), \quad |f(t, x) - f(t, y)| \leq F_\alpha |x - y|^\alpha,$$

whenever finite $t \in (T, S)$ and $x, y \in \mathbb{R}^d$ are such that $|x - y| \leq 1$.

Theorem 2.8. *Assume that we are given a function $g \in C^{2+\alpha}$. Then there exists a unique $u \in \mathcal{C}^{2+\alpha}(T, S)$ satisfying (2.7) and such that $u(S, x) = g(x)$. Moreover, for this solution, for any finite $t \in [T, S]$, we have*

$$\|u(t, \cdot)\|_{2+\alpha} \leq N(F_0 + F_\alpha + \|g\|_{2+\alpha}), \tag{2.8}$$

where $N = N(\delta, \alpha, d, K)$.

Proof. Uniqueness obviously follows from Theorem 2.4. To prove existence, introduce $\tilde{f}(t, x) = f(t, x)$ for $t \leq S$ and

$$\tilde{f}(t, x) = (\Delta + \partial/\partial t - \delta)g(x) = \Delta g(x) - \delta g(x),$$

for $t > S$. Also introduce the operator $\tilde{L}$ such that it coincides with L for $t \leq S$ and with $\Delta + \partial/\partial t - \delta$ for $t > S$.

Then by Theorem 2.5 there is a unique $u \in \mathcal{C}^{2+\alpha}(T)$ such that

$$u_t + \tilde{L}u = \tilde{f} \tag{2.9}$$

in $\mathbb{R}^{d+1}_T$. Obviously $g(x)$ is of class $\mathcal{C}^{2+\alpha}(S)$ and satisfies (2.9) in $\mathbb{R}^{d+1}_S$. By Theorem 2.5 we conclude that $u(t, x) = g(x)$ for $t \geq S$. In particular, $u \in \mathcal{C}^{2+\alpha}(T, S)$ and $u(S, x) = g(x)$.

Finally, estimate (2.8) follows immediately from (2.5) and the definition of $\tilde{f}$. The theorem is proved.

Remark 2.9. The previous theorem can be adapted to the case that $c(t,x)$ is only a *nonnegative* function on $\mathbb{R}^{d+1}_T$ with the other assumptions in Theorem 2.8 unchanged apart from (2.2) in Hypothesis 2.2 where we replace c with 1.

Indeed, if we want to solve equation (2.7) with $c(t,x) \geq 0$ and final condition g, we can first solve the Cauchy problem

$$u_t(t,x) + Lu(t,x) - u(t,x) = e^{t-S} f(t,x) \tag{2.10}$$

in $(T,S)\times\mathbb{R}^d$ with $u(S,x) = g(x)$, and then define $v(t,x) = e^{S-t}u(t,x)$, for $t \leq S$ and $v(t,x) = g(x)$, $t > S$. Clearly, $v \in \mathcal{C}^{2+\alpha}(T',S)$ for any finite $T' \in [T,S)$, $v(S,x) = g(x)$, and v solves (2.7). By Theorem 2.8, we get, for any finite $t \in [T,S]$,

$$\|u(t,\cdot)\|_{2+\alpha} \leq N(F_0 + F_\alpha + \|g\|_{2+\alpha}),$$

$$\|v(t,\cdot)\|_{2+\alpha} \leq e^{S-t} N(F_0 + F_\alpha + \|g\|_{2+\alpha}).$$

Uniqueness for (2.7) with $c(t,x) \geq 0$ follows easily from the uniqueness already proved for equation (2.10).

Remark 2.10. When the operator L has coefficients independent of t and $f = 0$, Theorem 2.8, in an obvious way (considering $S = 0$, $T = -\infty$ and inverting time) allows one to introduce the corresponding diffusion semigroup T_t of bounded operators mapping $C^{2+\alpha}$ into $C^{2+\alpha}$. By the maximum principle we have

$$\|T_t g\|_0 \leq \|g\|_0, \quad t \geq 0, \quad g \in C^{2+\alpha},$$

and so an approximation argument allows one to extend T_t from mappings $C^{2+\alpha} \to C^{2+\alpha}$ to mappings $UCB(\mathbb{R}^d) \to UCB(\mathbb{R}^d)$ (where $UCB(\mathbb{R}^d)$ stands for the Banach space of all real uniformly continuous and bounded functions defined on $\mathbb{R}^d$, endowed with the supremum norm). Moreover, by interpolation theorems, T_t will form a semigroup of bounded operators mapping C^α into C^α.

Several properties of the diffusion semigroup T_t, corresponding to an operator L with possibly unbounded time-independent coefficients a, b and c are investigated both from an analytic point of view (see [3] [4], [7], [18], [19], [20]) and from a probabilistic point of view (see [5], [6], [9], and the references therein).

3. Schauder estimates for equations with coefficients independent of x

In this section we concentrate on equations with the operator

$$L_0 u = a^{ij}(t) u_{x^i x^j},$$

where $(a^{ij}(t))$ is a symmetric matrix depending only on $t \in (T,\infty)$ in a measurable way and such that

$$K(\delta^{ij}) \geq (a^{ij}(t)) \geq \delta(\delta^{ij}), \quad t > T,$$

For $t > s$ set

$$A_{st} := \int_s^t a(r)\,dr, \quad B_{st} := A_{st}^{-1}.$$

Observe that the matrices A_{st} are nondegenerate so that B_{st} is well defined and

$$K(t-s)|\xi|^2 \geq A_{st}^{ij}\xi^i\xi^j \geq \delta(t-s)|\xi|^2,$$
$$\delta^{-1}(t-s)^{-1}|\xi|^2 \geq B_{st}^{ij}\xi^i\xi^j \geq K^{-1}(t-s)^{-1}|\xi|^2, \quad t > s, \;\; \xi \in \mathbb{R}^d.$$

Define ($I_{t>s}$ stands for the indicator function of the set $\{t > s\}$)

$$p(s,t,x) = I_{t>s}(4\pi)^{-d/2}(\det B_{st})^{1/2}\exp(-(B_{st}x,x)/4), \tag{3.1}$$
$$Gf(s,x) = \int_s^\infty \int_{\mathbb{R}^d} p(s,t,x-y)f(t,y)\,dydt$$
$$= \int_0^\infty \int_{\mathbb{R}^d} p(s,r+s,x-y)f(r+s,y)\,dydr,$$

for any bounded measurable function f on $\mathbb{R}^{d+1}$ with compact support.

The next two lemmas concerning the potential G are essentially known even if in the literature they are stated in a slightly different form (see [2], [10], [13], [16], [17]). For the sake of completeness we still give the proofs albeit sometimes somewhat sketchy.

Lemma 3.1. *If $u \in \mathcal{C}_0^{2+\alpha}(T)$, then for any finite $t \geq T$ and $x \in \mathbb{R}^d$ we have*

$$u(t,x) = -G(u_t + L_0u)(t,x). \tag{3.2}$$

Proof. Set $f = u_t + L_0u$ and observe that f is a bounded measurable function defined in $\mathbb{R}^{d+1}_T$ vanishing for large $|t| + |x|$. We have, for any finite $t > s \geq T$, $x \in \mathbb{R}^d$,

$$u(t,x) - u(s,x) = \int_s^t f(r,x)dr - \int_s^t L_0u(r,x)dr.$$

Using the Fourier transform in the space variable x we get

$$\hat{u}(t,\xi) - \hat{u}(s,\xi) = \int_s^t \hat{f}(r,\xi)dr + \int_s^t \xi^i\xi^j a^{ij}(r)\hat{u}(r,\xi)dr,$$

i.e., fixing $\xi \in \mathbb{R}^d$, we have a.s. in t, $\hat{u}_t(t,\xi) = \hat{f}(t,\xi) + \xi^i\xi^j a^{ij}(t)\hat{u}(t,\xi)$. It follows that

$$\hat{u}(t,\xi) = -\int_t^\infty e^{-A_{tr}^{ij}\xi^i\xi^j}\hat{f}(r,\xi)dr.$$

Taking the anti-Fourier transform we easily get the assertion.

Lemma 3.2. *If $f(t,x)$ is a bounded measurable function in $\mathbb{R}^{d+1}$ vanishing if $t \geq S$, for some (finite) constant S, and such that*

$$\sup_{t>T}[f(t,\cdot)]_\alpha < \infty, \tag{3.3}$$

then, for each finite $t \geq T$, the function $Gf(t,x)$ is twice continuously differentiable in x and, for any $x, y \in \mathbb{R}^d$, we have

$$|D^2 Gf(t,x)| \leq N \sup_{s>t}[f(s,\cdot)]_\alpha, \tag{3.4}$$

where N depends only on $(S-t)_+, d, \delta$, and α, and

$$|D^2 Gf(t,x) - D^2 Gf(t,y)| \leq N \sup_{s>t}[f(s,\cdot)]_\alpha |x-y|^\alpha, \tag{3.5}$$

where N depends only on d, δ, and α.

Proof. Apart from the fact that the constants N are independent of K, this is a standard result by now (it was first proved in [2] when a^{ij} are independent on t). It can also be proved by adapting the computations in Section IV.2 of [15] (see, in particular, pages 276 and 277 where the case of a^{ij} independent on t is treated) or, in a more direct way, arguing as in the proof of Theorem 4.2 in [22].

Estimates (3.4) and (3.5) are proved in Lemma 3.2 of [13] for measurable a^{ij} in a slightly sharper form (with the maximal function of $[f(s,\cdot)]_\alpha$ in place of $\sup_{s>t}$, which allowed one to consider spaces with norms that are C^α in x and L_p in t). However, f in [13] was assumed to be in $C_0^\infty(\mathbb{R}^{d+1})$.

Actually, in the proof of Lemma 3.2 of [13] the facts that f is so much regular in t and vanishes for large $|x|$ were never used and what was used is that f has compact support in t, is bounded, and the derivative of $f(s,x)$ in x up to the third order are continuous in x and bounded in (s,x). To relax further these requirements and include the class of f under consideration we introduce $f^{(\varepsilon)}(t,x) = (f(t,\cdot) * \zeta_\varepsilon)(x)$, where $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$ and ζ is a nonnegative C_0^∞ function on $\mathbb{R}^d$ which integrates to one. Owing to (3.3) we have

$$\|f - f^{(\varepsilon)}\|_{0,T} \leq N\varepsilon^\alpha, \quad \|Gf - Gf^{(\varepsilon)}\|_{0,T} \leq N\varepsilon^\alpha,$$

$$\sup_{s>t}[f^{(\varepsilon)}(s,\cdot)]_\alpha \leq \sup_{s>t}[f(s,\cdot)]_\alpha.$$

Hence, for each t, as $\varepsilon \to 0$, the functions $Gf^{(\varepsilon)}(t,\cdot)$ converge to $Gf(t,\cdot)$ uniformly and, owing to (3.4) and (3.5), have uniformly bounded and uniformly continuous second-order derivatives in x. A standard result from Calculus implies that, for each t, $Gf(t,x)$ is twice continuously differentiable in x and $D^2Gf^{(\varepsilon)}(t,\cdot) \to D^2Gf(t,\cdot)$. By passing to the limit in (3.4) and (3.5) with $f^{(\varepsilon)}$ in place of f we obtain the desired result. This will again prove our estimates with constants depending on K. The fact, which will not be used in this article, that the constants N are independent on K follows from a general result proved in [14].

We need a result on the unique solvability of the heat equation for which we could not find a precise reference in the literature. A slight difficulty in proving it is caused by the fact that the datum f is only measurable in time (indeed, if f is continuous also in time, the proof is straightforward and the result becomes well known).

Lemma 3.3. *Let $T > -\infty$ and assume that $f(t,x)$ is a bounded measurable function vanishing for $t \geq S$, where S is a constant, and such that* (3.3) *holds. Then there exists a unique function $u \in \mathcal{C}^{2+\alpha}(T)$ such that in $\mathbb{R}^{d+1}_T$ we have*

$$u_t + \Delta u - \delta u = f. \tag{3.6}$$

Furthermore, $u(t,x) = 0$ for $t \geq S$.

Proof. Uniqueness follows from Theorem 4.1, which we prove later in a much more general setting.

To prove existence set $g(t,x) = -e^{\delta t} f(t,x)$ and make the change of the unknown function $v(t,x) = e^{\delta t} u(t,x)$. Then (3.6) becomes

$$v_t + \Delta v = -g. \tag{3.7}$$

Lemma 3.1 suggests a natural candidate for v:

$$v(t,x) = G_0 g(t,x),$$

where by G_0 we denote the operator G constructed from $a^{ij} \equiv \delta^{ij}$. Therefore, we introduce v by the above formula and proceed further.

Obviously $v(t,x)$ is bounded and vanishes for $t \geq S$. Owing to Lemma 3.2 and interpolation inequalities, $v \in \mathcal{C}^{2+\alpha}(T)$. By the way, this and the fact that $T > -\infty$ imply that $u(t,x) := e^{-\delta t} v(t,x)$ is of class $\mathcal{C}^{2+\alpha}(T)$ as well. Now the relation between (3.6) and (3.7) shows that to finish proving the lemma it only remains to prove that (3.7) holds.

First, observe that in our case $p(t,s,x) = p(0,s-t,x) =: p(t-s,x)$ (see (3.1)) and define the heat semigroup by

$$T_t h(x) = (p(t,\cdot) * h)(x), \quad t > 0$$

acting on bounded measurable functions h defined on $\mathbb{R}^d$. Then, take an $\varepsilon>0$ and introduce $g^{(\varepsilon)}(t,\cdot)=T_\varepsilon g(t,\cdot)$,

$$v^{(\varepsilon)}(t,x)=G_0g^{(\varepsilon)}(t,x)=\int_t^S T_{s-t}T_\varepsilon g(s,x)\,ds=\int_t^S T_{s-t+\varepsilon}g(s,x)\,ds. \tag{3.8}$$

One knows that, for each x and bounded h, the function $T_\tau h(x)$ as a function of τ is continuously differentiable in τ for $\tau>0$, infinitely differentiable in x, and

$$\frac{\partial}{\partial\tau}T_\tau h(x)=\Delta T_\tau h(x).$$

In particular, for any τ the rules of differentiating integrals depending on parameters imply that

$$\frac{\partial}{\partial s}T_{s-t+\varepsilon}g(\tau,x)=\Delta T_{s-t}T_\varepsilon g(\tau,x)=T_{s-t}\Delta g^{(\varepsilon)}(\tau,x)$$

for $s>t$. It follows that

$$T_{s-t+\varepsilon}g(\tau,x)=T_\varepsilon g(\tau,x)+\int_t^s\frac{\partial}{\partial r}T_{r-t+\varepsilon}g(\tau,x)\,dr$$

$$=T_\varepsilon g(\tau,x)+\int_t^s T_{r-t}\Delta g^{(\varepsilon)}(\tau,x)\,dr.$$

We plug in here s in place of τ and go back to (3.8). Then we find

$$v^{(\varepsilon)}(t,x)=\int_t^S g^{(\varepsilon)}(s,x)\,ds+\int_t^S\int_t^s T_{r-t}\Delta g^{(\varepsilon)}(s,x)\,drds. \tag{3.9}$$

Changing the variable r by $\tau=s-r+t$ and then using Fubini's theorem we obtain

$$\int_t^S\int_t^s T_{r-t}\Delta g^{(\varepsilon)}(s,x)\,drds=\int_t^S\int_t^s T_{s-\tau}\Delta g^{(\varepsilon)}(s,x)\,d\tau ds$$

$$=\int_t^S\Big(\int_\tau^S T_{s-\tau}\Delta g^{(\varepsilon)}(s,x)\,ds\Big)d\tau=\int_t^S\Delta v^{(\varepsilon)}(\tau,x)\,d\tau.$$

Upon combining this with (3.9) we conclude that $v^{(\varepsilon)}$ satisfies (3.7) with $g^{(\varepsilon)}$ in place of g.

Finally, we send $\varepsilon\downarrow 0$ and observe that $g^{(\varepsilon)}\to g$ uniformly in $\mathbb{R}^{d+1}_T$ since g is uniformly continuous in x. It follows that $v^{(\varepsilon)}\to v$ uniformly in $\mathbb{R}^{d+1}_T$. Furthermore, $D^2G_0g^{(\varepsilon)}\to D^2G_0g$ uniformly in $\mathbb{R}^{d+1}_T$ since for any $\beta\in(0,\alpha)$, by (3.4), we have

$$\|D^2G_0(g^{(\varepsilon)}-g)\|_{0,T}\le N\sup_{s>T}[g^{(\varepsilon)}(s,\cdot)-g(s,\cdot)]_\beta\to 0,$$

as $\varepsilon \downarrow 0$, where the last relation follows from the fact that (3.3) holds with g in place of f (just in case, note that N depends on $(S-T)_+$). This argument allows us to pass to the limit in the integral version of (3.7). The lemma is proved.

The next elementary result is well known.

Lemma 3.4. *Let $\gamma \geq 1$ and let Q be a convex closed round cone in $\mathbb{R}^d$ with vertex at the origin such that for any unit ball which lie inside Q the distance of its center to the origin is greater than or equal to γ. Let (u^{ij}) be a $d \times d$ symmetric matrix. Then there is a constant $N = N(\gamma, d)$ such that for any $i, j = 1, ..., d$ we have*

$$|u^{ij}| \leq N \max_{|\xi|=1, \xi \in Q} \Big| \sum_{i,j=1}^{d} u^{ij} \xi^i \xi^j \Big|.$$

Now follows an embedding result generalizing [10, Lemma 1]. Its proof uses parabolic dilations and is simpler than the one in [10]. Its generality is in part motivated by possible applications to boundary value problems.

Take γ and Q as in Lemma 3.4, take an $h > 0$, and consider the truncated cone

$$Q_h = Q \cap \{x : |x| \leq h\}.$$

The spaces $C^\alpha(Q_h)$ are defined in the same way as $C^\alpha = C^\alpha(\mathbb{R}^d)$. We also write $[\cdot]_{\alpha,Q_h}$ to denote the usual Hölder seminorm in $C^\alpha(Q_h)$. Similarly we introduce the spaces $C^{2+\alpha}(Q_h)$ and the seminorms $[\cdot]_{2+\alpha,Q_h}$. Notice that Q_h is a closed set. In particular, the functions from $C^{2+\alpha}(Q_h)$ are twice continuously differentiable in the interior of Q_h and their derivatives admit continuous extension to the boundary of Q_h. In the following lemma by $D^2u(r, 0)$ and $Du(r, 0)$ we mean these continuations.

Lemma 3.5. *Let $u : [0, h^2] \times Q_h \to \mathbb{R}$ be a continuous function such that $u(t, \cdot) \in C^{2+\alpha}(Q_h)$, $t \in [0, h^2]$, and assume that there exists a function $g(t, x)$ defined on $[0, h^2] \times Q_h$ such that $g(t, \cdot) \in C^\alpha(Q_h)$, $t \in [0, h^2]$, and (2.3) is verified for $0 \leq s \leq t \leq h^2$, $x \in Q_h$ (we set $u_t = g$).*

Then there is a constant $N = N(\gamma, d)$ such that

$$|D^2u(h^2, 0) - D^2u(0, 0)| \leq N I_h h^\alpha, \tag{3.10}$$

$$|Du(h^2, 0) - Du(0, 0)| \leq N I_h h^{1+\alpha}, \tag{3.11}$$

where

$$I_h := \sup_{r \in [0,h^2]} \big([u_t(r, \cdot)]_{\alpha,Q_h} + [D^2u(r, \cdot)]_{\alpha,Q_h} \big).$$

Proof. First we deal with (3.10). The parabolic dilation $(s,x)\mapsto(4^{-1}h^{-2}s,2^{-1}h^{-1}x)$ allows us to assume that $h=2$.

Next, assume that the first basis vector ℓ is inside Q_2 $(=Q_h)$. Write

$$|D_{11}u(4,0)-D_{11}u(0,0)|\leq|D_{11}u(4,0)-[u(4,2\ell)-2u(4,\ell)+u(4,0)]|$$
$$+|D_{11}u(0,0)-[u(0,2\ell)-2u(0,\ell)+u(0,0)]|+I_++I_-,$$

where

$$I_\pm=|[u(4,\ell\pm\ell)-u(0,\ell\pm\ell)]-[u(4,\ell)-u(0,\ell)]|.$$

Taylor's formula shows that

$$|D_{11}u(4,0)-[u(4,2\ell)-2u(4,\ell)+u(4,0)]|\leq 2[D_{11}u(4,\cdot)]_{\alpha,Q_2},$$
$$|D_{11}u(0,0)-[u(0,2\ell)-2u(0,\ell)+u(0,0)]|\leq 2[D_{11}u(0,\cdot)]_{\alpha,Q_2}.$$

By the Newton-Leibnitz formula

$$I_\pm=\int_0^4[u_t(r,\ell\pm\ell)-u_t(r,\ell)]\,dt,$$

which implies that

$$I_\pm\leq\sup_{r\in[0,4]}[u_t(r,\cdot)]_{\alpha,Q_2}.$$

Upon combining the above estimates, we come to

$$|D_{11}u(4,0)-D_{11}u(0,0)|\leq 4I_2.$$

Having this estimate proved for $\xi^i\xi^jD_{ij}u$ with ξ being the first basis vector, under the assumption that it lies inside Q_2, we also have

$$|\xi^i\xi^jD_{ij}u(4,0)-\xi^i\xi^jD_{ij}u(0,0)|\leq 4I_2$$

for all unit $\xi\in Q_2$. Applying Lemma 3.4 yields (3.10).

In light of (3.10), estimate (3.11) is, actually, classical and we give its proof just for completeness. Again we may assume that $h=2$ and first consider the case when $\ell\in Q_2$. Then

$$|D_1u(4,0)-D_1u(0,0)|\leq|J_1(1)|+|J_2|,$$

where

$$J_1(r)=[u(4,r\ell)-u(4,0)-rD_1u(4,0)]-[u(0,r\ell)-u(0,0)-rD_1u(0,0)],$$

$$J_2=[u(4,\ell)-u(4,0)]-[u(0,\ell)-u(0,0)]=\int_0^4[u_t(r,\ell)-u_t(r,0)]\,dr.$$

We estimate $|J_2|$ in the same way as $I_\pm$ above and notice that by Taylor's formula (or by integrating by parts)

$$J_1(r)=\int_0^r(r-z)[D_{11}u(4,z\ell)-D_{11}u(0,z\ell)]\,dz.$$

We can certainly take the point $z\ell$ as the origin in $\mathbb{R}^d$ and use (3.10). Then we see that $|J_1(1)| \leq NI$ and this leads to the estimate

$$|\xi^i D_i u(4,0) - \xi^i D_i u(0,0)| \leq NI$$

if $\xi = \ell$. The same estimate holds for any unit vector $\xi \in Q_2$ and this implies (3.11). The lemma is proved.

By shifting the origin in $\mathbb{R}^{d+1}$ to points (s,x) and denoting $t-s=h^2$ we obtain the following.

Corollary 3.6. *For any $u \in \mathcal{C}^{2+\alpha}(T)$ and finite $t,s \geq T$ and $x \in \mathbb{R}^d$ we have*

$$\begin{gathered}|D^2u(t,x) - D^2u(s,x)| \leq NI|t-s|^{\alpha/2}, \qquad (3.12)\\ |Du(t,x) - Du(s,x)| \leq NI|t-s|^{(1+\alpha)/2},\end{gathered}$$

where N depends only on d and

$$I = \sup_{r\in[s,t]} \big([u_t(r,\cdot)]_\alpha + [u(r,\cdot)]_{2+\alpha}\big).$$

Remark 3.7. In connection with Corollary 3.6 it is also worth noting that if $u \in \mathcal{C}^{2+\alpha}(T)$, then u is locally Lipschitz in $(t,x) \in \mathbb{R}^{d+1}_T$ since the derivative Du is bounded and u_t is locally bounded.

Lemma 3.8. *If $u \in \mathcal{C}_0^{2+\alpha}(T)$, then for any $x \in \mathbb{R}^d$ and finite $t,s \geq T$ we have*

$$[u(t,\cdot)]_{2+\alpha} \leq N \sup_{r>t}[(u_t + L_0u)(r,\cdot)]_\alpha, \qquad (3.13)$$

$$|D^2u(s,x) - D^2u(t,x)| \leq N \sup_{r>t}[(u_t + L_0u)(r,\cdot)]_\alpha |s-t|^{\alpha/2}, \qquad (3.14)$$

where the constant N depends only on d,δ,α (and is independent of K).

To prove the lemma it suffices to make just few comments. Indeed, (3.13) follows directly from Lemmas 3.1 and 3.2. Estimate (3.14) with f in place of $u_t + L_0u$ follows from Theorem 3.3 of [13] for u in the form Gf if $f \in C_0^\infty(\mathbb{R}^{d+1})$. By Lemma 3.1 any $u \in \mathcal{C}_0^{2+\alpha}(T)$ has this form with f having less regularity than it is required in [13]. Then one can repeat what is said concerning the proof of Lemma 3.2.

Remark 3.9. It is also worth noting that (3.14) will not be used in the future and if one does not care about the statement that N in (3.14) is independent of K, then one can get the estimate from Corollary 3.6 and (3.13) since

$$\begin{aligned}[u_t(r,\cdot)]_\alpha &\leq [(u_t + L_0u)(r,\cdot)]_\alpha + [L_0u(r,\cdot)]_\alpha\\ &\leq [(u_t + L_0u)(r,\cdot)]_\alpha + N[u(r,\cdot)]_{2+\alpha},\end{aligned}$$

where $N = N(K,d)$. One can also note that, in turn, (3.12) follows from (3.14), which is seen if one takes $L_0 = \Delta$.

Here is a rather surprising generalization of estimate (3.13).

Lemma 3.10. *Let N be the constant from* (3.13).

(i) Let $u \in \mathcal{C}^{2+\alpha}(T)$ be such that for any $\zeta \in C_0^\infty(\mathbb{R})$ the function $\zeta(t)u(t,x)$ belongs to $\mathcal{C}_0^{2+\alpha}(T)$. Then for any finite $t \geq T$, any locally bounded, nonnegative, measurable function $c_0(t)$, and any locally bounded, $\mathbb{R}^d$-valued, measurable function $b_0(t)$ we have

$$[u(t,\cdot)]_{2+\alpha} \leq N \sup_{s>t} [(u_t + L_0 u + b_0^i u_{x^i} - c_0 u)(s,\cdot)]_\alpha. \tag{3.15}$$

(ii) If the coefficients of L_0 are independent of t, then for any $u \in C^{2+\alpha}$ with compact support, any constant $c_0 \geq 0$, and any constant vector $b_0 \in \mathbb{R}^d$, we have

$$[u]_{2+\alpha} \leq N[L_0 u + b_0^i u_{x^i} - c_0 u]_\alpha.$$

Proof. Assertion (ii) follows directly from (i) if we take in the latter u independent of t.

We prove (i) first assuming that $u \in \mathcal{C}_0^{2+\alpha}(T)$. For $t \in \mathbb{R}$ define

$$B(t) = \int_0^t b_0(s)\, ds, \quad v(t,x) = u(t, x + B(t)).$$

Since $u \in \mathcal{C}_0^{2+\alpha}(T)$ and the derivative of B is locally bounded, $v \in \mathcal{C}_0^{2+\alpha}(T)$. By plugging in v in place of u in (3.13) we obtain (3.15) if $c_0 \equiv 0$.

To let c_0 enter into the picture introduce

$$C(t) = \int_0^t c_0(s)\, ds, \quad v(t,x) = e^{-C(t)} u(t,x).$$

Again $v \in \mathcal{C}_0^{2+\alpha}(T)$ and by substituting v in place of u in (3.15) with $c_0 \equiv 0$ we get

$$[u(t,\cdot)]_{2+\alpha} e^{-C(t)} \leq N \sup_{s>t} [(u_t + L_0 u + b_0^i u_{x^i} - c_0 u)(s,\cdot)]_\alpha e^{-C(s)},$$

$$[u(t,\cdot)]_{2+\alpha} \leq N \sup_{s>t} [(u_t + L_0 u + b_0^i u_{x^i} - c_0 u)(s,\cdot)]_\alpha e^{C(t)-C(s)}.$$

In case $u \in \mathcal{C}_0^{2+\alpha}(T)$, this yields (3.15) in full generality since $C(s) \geq C(t)$ for $s > t$.

To pass to the case of general u, take a $\zeta \in C_0^\infty(\mathbb{R})$, such that $\zeta(0) = 1$ and $0 \leq \zeta \leq 1$, and substitute

$$u^n(s,x) := \zeta(s/n) u(s,x)$$

in (3.15). Then let $n \to \infty$. After that it will only remain to observe that

$$[(u_t^n + L_0 u^n + b_0^i u_{x^i}^n - c_0 u^n)(s,\cdot)]_\alpha \leq n^{-1} [u(s,\cdot)]_\alpha \sup |\zeta'|$$

$$+[u_t + L_0 u + b_0^i u_{x^i} - c_0 u)(s,\cdot)]_\alpha.$$

The lemma is proved.

4. Proof of the main results

We start with proving the following a priori estimate of the kind of the maximum principle, in which the assumptions on the coefficients are weaker than Hypotheses 2.1 and 2.2.

The proof of the next result seems to be new even in the case of bounded coefficients a, b, c and bounded datum f. It is important for the future to observe that condition (iii) of Theorem 4.1 below is satisfied for any $u \in \mathcal{C}^{2+\alpha}(T)$, which follows from Corollary 3.6 and Remark 3.7 applied to $u(t,x)\zeta(x)$, where ζ is any function of class $C_0^\infty(\mathbb{R}^d)$. It is also worth noting that the result of Theorem 4.1 cannot be obtained from the Alexandrov maximum principle for parabolic equations since it requires $u \in W^{1,2}_{d+1,loc}(\mathbb{R}^{d+1}_T)$ and in the theorem u_t may be only locally summable to the first power. However, by using parabolic Aleksandrov estimates one could considerably relax the assumptions on u if c and f are locally bounded.

Theorem 4.1. *(i) Assume that in $\mathbb{R}^{d+1}_T$ we have an operator L as in (1.1) and f such that a^{ij}, b^i, c, f are measurable functions, $(a^{ij}(t,x))$ is symmetric and nonnegative, $c(t,x) \geq \delta$, $-f(t,x) \leq F_0 c(t,x)$,*

$$|a(t,x)| \leq K(t)(1+|x|^2), \quad |b(t,x)| \leq K(t)(1+|x|)$$

in $\mathbb{R}^{d+1}_T$, where $K(t)$ is a locally bounded function on $\mathbb{R}$;

(ii) Assume that, for any x, the functions $c(t,x)$ and $f(t,x)$ are locally integrable in t on $\mathbb{R}$;

(iii) Assume that in $\mathbb{R}^{d+1}_T$ we are given a bounded continuous function $u(t,x)$ which is twice continuously differentiable in x for each finite $t \geq T$ and such that u_x and u_{xx} are continuous with respect to (t,x) in $\mathbb{R}^{d+1}_T$;

(iv) Finally, assume that (2.4) holds for each $x \in \mathbb{R}^d$ and finite $t > s$ such that $s \geq T$. Then in $\mathbb{R}^{d+1}_T$

$$u(t,x) \leq F_0. \tag{4.1}$$

Furthermore, if $|f(t,x)| \leq F_0 c(t,x)$, then $|u(t,x)| \leq F_0$ in $\mathbb{R}^{d+1}_T$.

Proof. Obviously, the second assertions follows from the first one. To prove the first assertion observe that the function $v = u - F_0$ satisfies (1.2) with f replaced by $g = cF_0 + f \geq 0$ and $-g \leq 0 \cdot c$. If our result is true for $f \geq 0$ and $F_0 = 0$, then $v \leq 0$ and $u \leq F_0$. It follows that in the rest of the proof we may confine ourselves to the case that $F_0 = 0$.

Without losing generality we may also assume that $T > -\infty$ and even that $T = 0$.

Next notice that to prove (4.1) (with $F_0 = 0$), it suffices to prove that for any $S > 0$, $t \in [0, S]$, and $x \in \mathbb{R}^d$ we have

$$u(t,x) \le e^{\delta'(t-S)} \sup_{y\in\mathbb{R}^d} u_+(S,y), \tag{4.2}$$

where $\delta' = \delta/2$. Indeed, one can then let $S \to \infty$ and use that u is bounded by assumption. In turn, to prove (4.2) it suffices to show that for any $\gamma > 0$

$$\bar u(t,x) := (u(t,x) - \gamma w(t,x))e^{-\delta' t} \le e^{-\delta' S} \sup_{y\in\mathbb{R}^d} u_+(S,y), \tag{4.3}$$

where

$$w(t,x) = (1 + |x|^2)e^{-N_0 t}$$

and we choose the constant N_0 so large that in $[0, S] \times \mathbb{R}^d$ we have

$$w_t + Lw =: g < 0,$$

which is clearly possible.

We use the fact that u is bounded and continuous, w tends to infinity as $|x| \to \infty$ to conclude that at certain point $(t_0, x_0) \in [0, S] \times \mathbb{R}^d$ the function $\bar u(t,x)$ takes its maximum value over $[0, S] \times \mathbb{R}^d$. If

$$\bar u(t_0,x_0) \le e^{-\delta' S} \sup_{y\in\mathbb{R}^d} u_+(S,y),$$

then (4.3) obviously holds. Therefore, it suffices to show that the inequality

$$\bar u(t_0,x_0) > e^{-\delta' S} \sup_{y\in\mathbb{R}^d} u_+(S,y), \tag{4.4}$$

is impossible.

We argue by contradiction and suppose that (4.4) holds. Then obviously $0 \le t_0 < S$ and $\bar u(t_0, x_0) > 0$. Furthermore, at (t_0, x_0) we have that $D\bar u = 0$ and the Hessian matrix $D^2\bar u$ is nonpositive, which implies that, for any $\tau > 0$, there exists a $\theta > 0$ such that

$$|D\bar u(t,x_0)| \le \tau, \quad D^2\bar u(t,x_0) \le \tau(\delta^{ij}), \quad \bar u(t,x_0) > 0$$

whenever $0 \le t - t_0 \le \theta$. Also

$$0 \ge \bar u(t,x) - \bar u(t_0,x_0) = \int_{t_0}^t \bar u_t(s,x_0)\,ds. \tag{4.5}$$

However,

$$\bar u_t = -\delta'\bar u + (u_t - \gamma w_t)e^{-\delta' t}$$
$$= -\delta'\bar u - L\bar u + fe^{-\delta' t} - \gamma g e^{-\delta' t} \ge -\delta'\bar u - L\bar u$$

and the last expression at points (t,x_0) such that $0\le t-t_0\le\theta$ is greater than

$$-N\tau+(c-\delta')\bar u\ge -N\tau+\delta'\bar u,$$

where the constant N is independent of t and τ. Hence, for $t_0+\theta\ge t\ge t_0$, equation (4.5) implies that

$$0\ge -N\tau(t-t_0)+\delta'\int_{t_0}^t \bar u(s,x_0)\,ds.$$

We divide both part of this inequality by $t-t_0$, let $t\downarrow t_0$, and use the continuity of $\bar u$. Then we conclude that $\delta'\bar u(t_0,x_0)\le N\tau$ and since $\tau>0$ is arbitrary, $\bar u(t_0,x_0)\le 0$, which yields the desired contradiction with (4.4). The theorem is proved.

Proof of Theorem 2.4. As usual the second assertion is obtained from the first one by taking there u independent of t.

While proving (i), first observe that it suffices to prove the estimate

$$\sup_{t\ge T}[u(t,\cdot)]_{2+\alpha}\le N(F_0+F_\alpha)+N\sup_{t\ge T}\|u(t,\cdot)\|_2. \tag{4.6}$$

Indeed, once (4.6) is proved, (2.5) follows easily from the interpolation inequality

$$\|v\|_2\le N(\varepsilon)\|v\|_0+\varepsilon[v]_{2+\alpha},\quad \varepsilon>0,\quad v\in C^{2+\alpha},$$

and the fact that $|u(t,x)|\le F_0$ in $\mathbb{R}^{d+1}_T$ by Theorem 4.1.

To prove (4.6) fix an $\varepsilon\in(0,1/2)$ and a $\zeta\in C_0^\infty(\mathbb{R}^d)$ with support in the ball of radius 2ε centered at the origin and such that $\zeta(x)=1$ for $|x|\le\varepsilon$. Also take a point $(t_0,x_0)\in\mathbb{R}^{d+1}_T$ and introduce $x(t)$ as a solution (not necessarily unique) of the problem

$$x(t)=x_0+\int_{t_0}^t b(s,x(s))\,ds,\quad t\in\mathbb{R},$$

where $b(t,x)$ is the vector with coordinates $b^i(t,x), i=1,\dots,d$. Such solution exists since b is locally bounded, continuous in x uniformly in t, and grows at most linearly in x.

Set

$$a_0^{ij}(t)=a^{ij}(t,x(t)),\quad b_0(t)=b(t,x(t)),\quad c_0(t)=c(t,x(t)),$$
$$L_0=a_0^{ij}(t)D_{ij}+b_0(t)D_i-c_0(t),$$
$$f_0(t)=f(t,x(t)),$$
$$u_0(t)=-\int_t^\infty f_0(s)\exp(-\int_t^s c_0(r)\,dr)\,ds,$$
$$\eta(t,x)=\zeta(x-x(t)),\quad v(t,x)=[u(t,x)-u_0(t)]\eta(t,x).$$

Observe that if $\eta(t,x)\ne0$, then $|x-x(t)|\le 2\varepsilon$, so that

$$|a^{ij}(t,x)-a^{ij}_0(t)|\le 2^\alpha K\varepsilon^\alpha,\quad |b(t,x)-b_0(t)|\le 2^\alpha K\varepsilon^\alpha d,$$
$$|c(t,x)-c_0(t)|\le 2^\alpha K\varepsilon^\alpha,\quad |f(t,x)-f_0(t)|\le 2^\alpha F_\alpha\varepsilon^\alpha. \tag{4.7}$$

Also

$$\eta_t(t,x)+b^i_0(t)\eta_{x_i}(t,x)=0,$$
$$u_{0t}+L_0u_0=f_0.$$

Next, by Lemma 3.10 applied to v, for $x\in\mathbb{R}^d$ such that $|x-x_0|\le\varepsilon$, we have $\eta(t_0,x)=1$ and

$$I:=\frac{|D^2u(t_0,x)-D^2u(t_0,x_0)|}{|x-x_0|^\alpha}=\frac{|D^2v(t_0,x)-D^2v(t_0,x_0)|}{|x-x_0|^\alpha}$$
$$\le N\sup_{s>t_0}[(v_t+L_0v)(s,\cdot)]_\alpha.$$

Here

$$v_t+L_0v=\eta(u_t+L_0u-f_0)+(u-u_0)(\eta_t+L_0\eta+c_0\eta)+2a^{ij}_0\eta_{x^i}u_{x^j}$$
$$=\eta(f-f_0)+\eta(L_0-L)u+(u-u_0)a^{ij}_0\eta_{x^ix^j}+2a^{ij}_0\eta_{x^i}u_{x^j}.$$

Since on the support of η we have (4.7), it is standard to see that

$$I\le N(\varepsilon)F_\alpha+N\varepsilon^\alpha\sup_{s>t_0}[u(s,\cdot)]_{2+\alpha}$$
$$+N(\varepsilon)\sup_{s>t_0}\|u(s,\cdot)\|_2+N(\varepsilon)\sup|u_0|,$$

where $N=N(d,\alpha,K,\delta)$ and $N(\varepsilon)=N(\varepsilon,d,\alpha,K,\delta)$. Due to the arbitrariness of x_0 and x and the fact that obviously $|u_0(t)|\le F_0$ in $\mathbb{R}$, we obtain

$$[u(t_0,\cdot)]_{2+\alpha}\le N(\varepsilon)(F_0+F_\alpha)$$
$$+N\varepsilon^\alpha\sup_{s>t_0}[u(s,\cdot)]_{2+\alpha}+N(\varepsilon)\sup_{s>t_0}\|u(s,\cdot)\|_2.$$

Upon taking the sup of both sides with respect to $t_0\ge T$ we conclude

$$\sup_{t\ge T}[u(t,\cdot)]_{2+\alpha}\le N(\varepsilon)(F_0+F_\alpha)$$
$$+N\varepsilon^\alpha\sup_{t\ge T}[u(t,\cdot)]_{2+\alpha}+N(\varepsilon)\sup_{t\ge T}\|u(t,\cdot)\|_2.$$

After choosing ε appropriately, we finally get (4.6). The theorem is proved.

Corollary 4.2. *Let T_1, T_2, and R be finite numbers such that $T \leq T_1 < T_2$ and $R > 0$. Then under Hypotheses 2.1 and 2.2 for any $u \in \mathcal{C}^{2+\alpha}(T)$ satisfying* (1.2) *we have:*

$$|t-s|^{-1}|u(s,x)-u(t,x)| + |t-s|^{-\frac{1+\alpha}{2}}|Du(s,x)-Du(t,x)| + |t-s|^{-\alpha/2}|D^2u(s,x)-D^2u(t,x)| \leq N, \tag{4.8}$$

whenever $|x| \leq R$ and $s,t \in [T_1,T_2]$, $t < s$, where N depends only on δ, α, d, K, F_0, F_α, R, and sup norms of $|b(t,0)|, c(t,0), |f(t,0)|$ over $[T_1,T_2]$.

Proof. Take $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ such that $\zeta(t,x) = 1$ if $t \in [T_1,T_2]$ and $|x| \leq R$ and observe that for $v := u\zeta$ we have

$$v_t = u\zeta_t - \zeta Lu + \zeta f.$$

By combining this with (2.5) we see that for any finite $t \geq T$

$$\|v_t(t,\cdot)\|_\alpha + \|v(t,\cdot)\|_{2+\alpha} \leq N,$$

where N is as in the statement of the corollary. After that we get our assertion from Corollary 3.6 and Remark 3.7.

The a priori estimate (2.5) and the solvability of the heat equation (Lemma 3.3) allow us to prove the following result by the method of continuity.

Lemma 4.3. *Under Hypotheses 2.1 and 2.2 additionally assume that b, c, and f are bounded. Then there exists a unique $u \in \mathcal{C}^{2+\alpha}(T)$ satisfying* (1.2).

Proof. Uniqueness follows from Theorem 2.4. In the proof of existence first assume that $T > -\infty$ and take a finite $S > T$. For S and T fixed introduce the space $\mathfrak{C}^{2+\alpha}$ as a subspace of $\mathcal{C}^{2+\alpha}(T)$ such that $u(t,x) = 0$ for $t \geq S$ and

$$\|u\|_{\mathfrak{C}^{2+\alpha}} := \sup_{t\in[T,S]} \|u_t(t,\cdot)\|_\alpha + \sup_{t\in[T,S]} \|u(t,\cdot)\|_{2+\alpha} < \infty.$$

Also let $\mathfrak{C}^\alpha$ be the set of measurable functions $g(t,x)$ such that $g(t,x) = 0$ for $t \geq S$ and

$$\|g\|_{\mathfrak{C}^\alpha} := \sup_{t\in[T,S]} \|g(t,\cdot)\|_\alpha < \infty.$$

One can easily check that $\mathfrak{C}^{2+\alpha}$ and $\mathfrak{C}^\alpha$ are Banach spaces.

For $\lambda \in [0,1]$ consider the following family of equations

$$u_t + [\lambda L + (1-\lambda)(\Delta - \delta)]u = g \tag{4.9}$$

in $\mathbb{R}^{d+1}_T$. We call a $\lambda \in [0,1]$ "good" if for any $g \in \mathfrak{C}^\alpha$ there is a unique solution $u \in \mathfrak{C}^{2+\alpha}$ of (4.9). Notice that if $u \in \mathcal{C}^{2+\alpha}(T)$ satisfies (4.9) with a $g \in \mathfrak{C}^\alpha$, then by Theorem 2.4 (with S in place of T) we have $u(t,x) = 0$ for $t \geq S$ and

$$\sup_{t \geq T} \|u(t,\cdot)\|_{2+\alpha} \leq N\|g\|_{\mathfrak{C}^\alpha}.$$

From equation (4.9) we get an estimate of u_t and conclude that

$$\|u\|_{\mathfrak{C}^{2+\alpha}} \leq N\|g\|_{\mathfrak{C}^\alpha}, \tag{4.10}$$

where N is independent of λ, g, T, and S. Furthermore, by Lemma 3.3 we have that 0 is a "good" point.

We now claim that all points of $[0,1]$ are "good". To prove the claim we take a "good" point λ_0 (say $\lambda_0 = 0$) and rewrite (4.9) as

$$u_t + [\lambda_0 L + (1-\lambda_0)\Delta]u = g + (\lambda - \lambda_0)(\Delta - L)u. \tag{4.11}$$

Now fix $g \in \mathfrak{C}^\alpha$ and define a mapping $\mathcal{R}$ which sends $v \in \mathfrak{C}^{2+\alpha}$ into the solution $u \in \mathfrak{C}^{2+\alpha}$ of the equation

$$u_t + [\lambda_0 L + (1-\lambda_0)\Delta]u = g + (\lambda - \lambda_0)(\Delta - L)v. \tag{4.12}$$

Observe that owing to our assumptions and the choice of λ_0 the right-hand side of (4.12) is in $\mathfrak{C}^\alpha$ and the mapping $\mathcal{R}$ is well defined.

Estimate (4.10) shows that for any $v, w \in \mathfrak{C}^{2+\alpha}$

$$\|\mathcal{R}v - \mathcal{R}w\|_{\mathfrak{C}^{2+\alpha}} \leq N|\lambda - \lambda_0|\|v - w\|_{\mathfrak{C}^{2+\alpha}}$$

with N independent of λ_0, v, and w. It follows that there is an $\varepsilon > 0$ such that for $|\lambda - \lambda_0| \leq \varepsilon$ the mapping $\mathcal{R}$ is a contraction in $\mathfrak{C}^{2+\alpha}$ and has a fixed point u which obviously satisfies (4.11) and (4.9). Therefore such λ's are "good", which certainly proves our claim.

Owing to the boundedness of f, we have $fI_{[T,S]} \in \mathfrak{C}^\alpha$, so that we now know that (1.2) has a unique solution $u^S \in \mathcal{C}^{2+\alpha}(T)$ if we take $f(t,x)I_{t \leq S}$ in place of f. By the above

$$\sup_{t \geq T} \|u^S(t,\cdot)\|_{2+\alpha} \leq N \sup_{t \geq T} \|f(t,\cdot)\|_\alpha, \tag{4.13}$$

with N independent of f, T, and S. This estimate and Corollary 4.2 show that the family u^S, u^S_x, u^S_{xx} is uniformly bounded and equicontinuous on any bounded subset of $\mathbb{R}^{d+1}_T$. By the Ascoli-Arzelà theorem there is a continuous function u on $\mathbb{R}^{d+1}_T$ having bounded and continuous derivatives with respect to x up to the second order and a sequence $S_n \to \infty$ such that

$$(u^{S_n}, u^{S_n}_x, u^{S_n}_{xx}) \to (u, u_x, u_{xx})$$

as $n \to \infty$, uniformly on bounded subsets of $\mathbb{R}^{d+1}_T$. Passing to the limit in the equations (of course, in the integral form, see (2.4)) corresponding to u^{S_n} shows that u satisfies (1.2). Furthermore, (4.13) implies that with the same N

$$\sup_{t\ge T}\|u(t,\cdot)\|_{2+\alpha}\le N\sup_{t\ge T}\|f(t,\cdot)\|_{\alpha}.$$

Hence, $u\in\mathcal{C}^{2+\alpha}(T)$ and we have proved the lemma if $T>-\infty$.

To consider the case that $T=-\infty$ we define u_n as solutions of class $\mathcal{C}^{2+\alpha}(-n)$ of (1.2) in $\mathbb{R}^{d+1}_{-n}$. Since the right-hand sides of the equations for u_m agree on $\mathbb{R}^{d+1}_{-n}$ for $m\ge n$ by Theorem 2.4 we have that $u_m=u_n$ on $\mathbb{R}^{d+1}_{-n}$. Therefore, the limit u of u_n as $n\to\infty$ exists and obeys estimate (2.5) for $t\ge -n$ with arbitrary n and thus for any t. This shows that $u\in\mathcal{C}^{2+\alpha}(T)$ and finishes the proof of the lemma.

Here is a classical result from the standard $C^{2+\alpha}$ theory of parabolic equations.

Corollary 4.4. *In addition to the assumptions of Lemma 4.3 suppose that a,b,c, and f are $\alpha/2$-Hölder continuous in t with constant independent of x. Let $u\in\mathcal{C}^{2+\alpha}(T)$ be the solution of* (1.2). *Then,*

(i) For any $x\in\mathbb{R}^d$, finite s,t, with $T\le s<t$, estimate (4.8) *holds with a constant N depending only on δ, α, d, K, F_α, and the sup norms of b, c, and f;*

(ii) The function u_t is bounded on $\mathbb{R}^{d+1}_T$, α-Hölder continuous in x with constant independent of t and $\alpha/2$-Hölder continuous in t with constant independent of x.

Indeed, in assertion (ii) the α-Hölder continuity in x of u_t is obtained directly from the equation even without requiring the Hölder continuity of the data in t.

The remaining assertions of the lemma follow directly from the equation due to Theorem 2.4, Corollary 3.6, and Remark 3.7.

One could easily provide estimates of the above mentioned Hölder constants. We leave this to the reader.

Proof of Theorem 2.5. As usual uniqueness follows from the a priori estimate (see Theorem 2.4). To prove existence we use truncations. Introduce $\chi_n(\tau)=(-n)\vee\tau\wedge n$, $n=1,2,...$, $b^i_n=\chi_n(b^i)$, $i=1,\dots,d$, and similarly introduce c_n and f_n. Since χ_n are Lipschitz continuous with constant one and, for any $\tau,t\ge 0$, we have $\chi_n(\tau t)\le\tau\chi_n(t)$, the truncated coefficients satisfy Hypothesis 2.1 with the same constants K,F_0, and F_α.

By Lemma 4.3 there exists a unique $u_n\in\mathcal{C}^{2+\alpha}(T)$ satisfying equation (1.2) with b_n,c_n, and f_n in place of b,c, and f, respectively. By Theorem 2.4

$$\sup_{t\ge T}\|u_n(t,\cdot)\|_{2+\alpha}\le N(F_0+F_\alpha), \tag{4.14}$$

where N is independent of n. This estimate and Corollary 4.2 imply that the family u_n, u_{nx}, u_{nxx} is uniformly bounded and equicontinuous in any bounded subset of $\mathbb{R}^{d+1}_T$.

Since, for any bounded set $\Gamma \subset \mathbb{R}^{d+1}_T$ there exists an m such that $(b_n, c_n, f_n) = (b, c, f)$ on Γ for all $n \geq m$, to prove the theorem it only remains to repeat the part of the proof of Lemma 4.3 concerning u^{S_n}. The theorem is proved.

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