Densities for Ornstein-Uhlenbeck processes with jumps

11 april 2008

Enrico Priola [1]

Dipartimento di Matematica, Università di Torino,

via Carlo Alberto 10, 10123, Torino, Italy.

e-mail enrico.priola@unito.it

Jerzy Zabczyk [2]

Instytut Matematyczny, Polskiej Akademii Nauk,

ul. Sniadeckich 8, 00-950, Warszawa, Poland.

e-mail zabczyk@impan.gov.pl

Mathematics Subject Classification (2000): 60H10, 60J75, 47D07.

Key words: Ornstein-Uhlenbeck processes, absolute continuity, Lévy processes.

Abstract: We consider an Ornstein-Uhlenbeck process with values in $\mathbb{R}^n$ driven by a Lévy process (Z_t) taking values in $\mathbb{R}^d$ with d *possibly smaller* than n. The Lévy noise can have a degenerate or even vanishing Gaussian component. Under a controllability rank condition and a mild assumption on the Lévy measure of (Z_t), we prove that the law of the Ornstein-Uhlenbeck process at any time $t > 0$ has a density on $\mathbb{R}^n$. Moreover, when the Lévy process is of α-stable type, $\alpha \in (0, 2)$, we show that such density is a C^∞-function.

[1] Supported by the Italian National Project MURST "Equazioni di Kolmogorov" and by the Polish Ministry of Science and Education project 1PO 3A 034 29 "Stochastic evolution equations with Lévy noise".

[2] Supported by the Polish Ministry of Science and Education project 1PO 3A 034 29 "Stochastic evolution equations with Lévy noise".

1 Introduction and statement of the main results

We study absolute continuity of the laws of a n-dimensional Ornstein-Uhlenbeck process (X_t^x), which solves the stochastic differential equation

$$dX_t = AX_t dt + BdZ_t, \quad X_0 = x \in \mathbb{R}^n. \tag{1.1}$$

Here (Z_t) is a given Lévy process, with values in $\mathbb{R}^d$, defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The dimension d might be different and also *smaller* than n. Let us recall that (Z_t) is a stochastic process having independent, time-homogeneous increments and *c*àdlàg trajectories, starting from 0 (see [1]). Moreover A is a real $n \times n$ matrix and B a real $n \times d$ matrix.

Ornstein-Uhlenbeck processes appear in many areas of science, for instance in physics (see [10] and the references therein) and in mathematical finance (see [2], [5], [6] and the references therein). Ornstein-Uhlenbeck processes with jumps have recently received much attention (see [26], [24], [23] and [19]).

In the paper we present two main results: one on existence of densities of (X_t^x), and the other on the regularity of such densities. Both theorems assume the following (controllability) *rank condition*

$$\text{Rank}\,[B, AB, \ldots, A^{n-1}B] = n. \tag{1.2}$$

Here $[B, AB, \ldots, A^{n-1}B]$ denotes the $n \times nd$ matrix, composed of matrices $B, \ldots, A^{n-1}B$, which corresponds to the linear mapping: $(u_0, \ldots, u_{n-1}) \mapsto Bu_0 + \ldots + A^{n-1}Bu_{n-1}$, from $\mathbb{R}^{nd}$ into $\mathbb{R}^n$. An interesting example of an Ornstein-Uhlenbeck process with degenerate noise satisfying the rank condition (with $d = 1$ and $n = 2$) is a solution of the equation

$$\begin{cases} X_t^1 = Z_t, \quad X_0^1 = x_0^1, \\ X_t^2 = x_0^1 t + \displaystyle\int_0^t Z_s ds + x_0^2, \qquad t \geq 0, \;\; x = (x_0^1, x_0^2) \in \mathbb{R}^2. \end{cases} \tag{1.3}$$

It is a generalization of a famous example due to Kolmogorov, in which (Z_t) was a real Wiener process. In [14] Kolmogorov showed that the law of the random variable (X_t^1, X_t^2) is absolutely continuous with respect to the Lebesgue measure, for any $t > 0$, $x \in \mathbb{R}^2$, and, in fact, its density is a C^∞-function on $\mathbb{R}^2$. This Gaussian example has also been considered by Hörmander in [11].

When the process (Z_t) is a standard d-dimensional Wiener process, it is known that X_t^x has a density if and only if the rank condition holds (see, e.g., [7] and [8]). Moreover under (1.2) the random variables X_t^x, $t > 0$, $x \in \mathbb{R}^n$, have C^∞-densities. This regularity result can be easily extended to the case when the Lévy process (Z_t) is given by a non-degenerate d-dimensional Wiener process with a drift plus an independent pure jump process (see Section 2). Indeed, in such case, X_t^x has two independent components, one of which is a Gaussian Ornstein-Uhlenbeck process at time t having a C^∞-density. Note that convolution of two Borel probability measures has a density as long as at least one of the two measures has a density (see [23, Lemma 27.1]).

However, the situation is less clear if the Gaussian component of (Z_t) degenerates or vanishes. In this paper we consider such case. Indeed, we formulate our mild assumptions for absolute continuity only in terms of the Lévy measure of (Z_t). Our main first theorem is the following one.

Theorem 1.1. *Assume the rank condition* (1.2). *Assume also that the Lévy measure ν of (Z_t) is infinite and that there exists $r > 0$ such that ν restricted to the ball $\{x \in \mathbb{R}^d : |x| \le r\}$ has a density with respect to the Lebesgue measure. Then, for any $t > 0$ and $x \in \mathbb{R}^n$, the law of X_t^x is absolutely continuous.*

It also turns out (see Proposition 2.1) that under the assumptions of the theorem, the Ornstein-Uhlenbeck process (X_t^x) is strong Feller. There are a number of papers dealing with the absolute continuity of laws of degenerate diffusion processes with jumps (see [4], [16], [18], [15] and [13]). They apply appropriate extensions of Malliavin calculus for jump processes assuming also the well-known Hörmander condition on commutators (which becomes the rank condition (1.2) for Ornstein-Uhlenbeck processes). In [4] it is assumed that the Lévy measure of (Z_t) has a sufficiently smooth density. In [16], [15], [18] and [13] α-stable type Lévy processes (Z_t) are considered. The very weak sufficient conditions for the absolute continuity of the laws of degenerate diffusions with jumps, formulated in Theorem 1.1, are new. Moreover, in the proof, we use analytical methods as well as control theoretic arguments.

To formulate our second theorem, concerned with existence of regular densities, we need a new hypothesis on the Lévy measure ν.

Hypothesis 1.2. *There exist $C > 0$ and $\alpha \in (0, 2)$, such that, for sufficiently small $r > 0$, the following estimate holds:*

$$\int_{\{z \in \mathbb{R}^d \,:\, |\langle z,h \rangle| \le r\}} \langle z, h\rangle^2 \, \nu(dz) \ge C\, r^{2-\alpha}, \quad h \in \mathbb{R}^d, \ \ \text{with } |h| = 1. \tag{1.4}$$

This condition was introduced in [18]. However, both [18] and [13] prove C^∞-regularity of densities of solutions of SDEs with jumps assuming a *strictly stronger* version of Hypothesis 1.2 in which the integral with respect to ν is taken over the smaller set $\{z \in \mathbb{R}^d \,:\, |z| \le r\}$. An interesting example of measure ν for which (1.4) holds but the stronger hypothesis is not verified is given in [18, Remark 1].

Clearly, if (Z_t) is a d-dimensional α-stable process which is rotation invariant (i.e., $\psi(h) = c_\alpha |h|^\alpha$, for $h \in \mathbb{R}^d$, $\alpha \in (0, 2)$, where c_α is a positive constant) then (1.4) holds. Thus our next theorem generalizes the Kolmogorov regularity result concerning (1.3) to the case when (Z_t) is a Lévy process of α-stable type.

Theorem 1.3. *Assume the rank condition* (1.2) *and Hypothesis 1.2. Then, at any time $t > 0$, $x \in \mathbb{R}^n$, the Ornstein-Uhlenbeck process (X_t^x) has a C^∞-density with all bounded derivatives. Moreover, for any $t > 0$, $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ Borel and bounded,*

$$\begin{aligned} \mathbb{E}[f(X_t^x)] &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(e^{tA}x + y) \Big(\int_{\mathbb{R}^n} e^{-i\langle y,h\rangle} \exp\Big(-\int_0^t \psi(B^* e^{sA^*} h) ds \Big) dh \Big) dy \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(z) \Big(\int_{\mathbb{R}^n} e^{-i\langle z,h\rangle} e^{i\langle e^{tA^*}h, x\rangle} \exp\Big(-\int_0^t \psi(B^* e^{sA^*} h) ds \Big) dh \Big) dz. \end{aligned} \tag{1.5}$$

2 Existence of densities

Consider the Ornstein-Uhlenbeck process introduced in (1.1). It is well known that this is given by

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A} B dZ_s = e^{tA}x + Y_t, \ \ t \ge 0, \ x \in \mathbb{R}^n, \tag{2.1}$$

where the stochastic convolution Y_t can be defined as a limit in probability of Riemann sums (see, for instance, [23, Section 17] and [24]).

The law μ_t^x of X_t^x has the characteristic function (or Fourier transform) $\hat{\mu}_t^x$,

$$\hat{\mu}_t^x(h) = e^{i\langle e^{tA}x,h\rangle}\hat{\mu}_t(h) = e^{i\langle e^{tA^*}h,x\rangle}\exp\Big(-\int_0^t \psi(B^* e^{sA^*}h)ds\Big), \quad h \in \mathbb{R}^n, \quad (2.2)$$

where μ_t denotes the law of Y_t and ψ is the exponent of (Z_t),

$$\mathbb{E}[e^{i\langle u,Z_t\rangle}] = e^{-t\psi(u)},\ u \in \mathbb{R}^d.$$

By $\langle\cdot,\cdot\rangle$ and $|\cdot|$ we indicate the inner product and the Euclidean norm in $\mathbb{R}^k$, $k \in \mathbb{N}$, respectively. Moreover B^* denotes the adjoint (or transposed) matrix of B.

Recall the Lévy-Khintchine representation for ψ,

$$\psi(s) = \frac{1}{2}\langle Qs,s\rangle - i\langle a,s\rangle - \int_{\mathbb{R}^d}\Big(e^{i\langle s,y\rangle} - 1 - i\langle s,y\rangle\, I_D\,(y)\Big)\nu(dy), \quad s \in \mathbb{R}^d, \quad (2.3)$$

where I_D is the indicator function of the ball $D = \{x \in \mathbb{R}^d : |x| \le 1\}$, Q is a symmetric $d\times d$ non-negative definite matrix, $a \in \mathbb{R}^d$, and ν is the Lévy measure of (Z_t). Thus ν is a σ-finite measure on $\mathbb{R}^d$, such that

$$\nu(\{0\}) = 0, \qquad \int_{\mathbb{R}^d}(1 \wedge |y|^2)\,\nu(dy) < \infty.$$

The triplet (Q,a,ν) which gives (2.3) is unique. According to (2.3), the process (Z_t) can be represented by the Lévy-Itô decomposition as

$$Z_t = at + RW_t + Z_t^0, \qquad t \ge 0, \quad (2.4)$$

where R is a $d\times d$ matrix such that $RR^* = Q$, (W_t) is a standard $\mathbb{R}^d$-valued Wiener process and (Z_t^0) is a Lévy jump process (see [1]). The processes (W_t) and (Z_t^0) are independent.

Let (P_t) be the transition semigroup determined by (X_t^x), i.e.,

$$P_t f(x) = \mathbb{E}[f(X_t^x)],\ \ t \ge 0,\ x \in \mathbb{R}^n,$$

$f \in B_b(\mathbb{R}^n)$, where $B_b(\mathbb{R}^n)$ denotes the space of all real Borel and bounded functions on $\mathbb{R}^n$. The semigroup (P_t) (or the process (X_t^x)) is called *strong Feller* if $P_t f$ is a continuous function, for any $t > 0$ and for any $f \in B_b(\mathbb{R}^n)$.

Applying a result due to Hawkes (see [12]) we show now that the strong Feller property for (P_t) is equivalent to the existence of a density for the law of X_t^x, for any $t > 0$, $x \in \mathbb{R}^n$. This result holds for any Ornstein-Uhlenbeck process defined in (1.1) (without requiring the rank condition (1.2)). For related results in infinite dimensions, see [22] and [20].

Proposition 2.1. *The semigroup (P_t) is strong Feller if and only if, for each $t > 0$, $x \in \mathbb{R}^n$, the law μ_t^x of X_t^x is absolutely continuous with respect to the Lebesgue measure.*

Proof. Fix $t > 0$ and let μ_t be the law of Y_t (see (2.1)). Since $\mu_t^x = \delta_{e^{tA}x} * \mu_t$ (where δ_a denotes the Dirac measure concentrated in $a \in \mathbb{R}^n$) μ_t^x is absolutely continuous, for any $x \in \mathbb{R}^n$, if and only if μ_t has the same property.

We write, for any $f \in B_b(\mathbb{R}^n)$, $x \in \mathbb{R}^n$,

$$P_t f(x) = \int_{\mathbb{R}^n} f(e^{tA}x + y)\mu_t(dy) = \int_{\mathbb{R}^n} (f \circ e^{tA})(x + e^{-tA}y)\mu_t(dy) = \int_{\mathbb{R}^n} (f \circ e^{tA})(x + z)(e^{-tA} \circ \mu_t)(dz),$$

where $(e^{-tA} \circ \mu_t)$ is the image of the probability measure μ_t under e^{-tA}. Applying [12, Lemma 2.1], we know that the Markov operator $T_t g(x) = \int_{\mathbb{R}^n} g(x+z)(e^{-tA} \circ \mu_t)(dz)$, $x \in \mathbb{R}^n$, maps Borel and bounded functions into continuous ones if and only if $(e^{-tA} \circ \mu_t)$ is absolutely continuous with respect to the Lebesgue measure. Hence $P_t f$ is continuous, for any $f \in B_b(\mathbb{R}^n)$, if and only if $(e^{-tA} \circ \mu_t)$ is absolutely continuous. This gives the assertion, since e^{tA} is an isomorphism. ∎

The proof of Theorem 1.1 requires two lemmas. The first one is of independent interest.

Lemma 2.2. *Assume the rank condition* (1.2). *Then there exists $T_0 > 0$ (depending on the dimension n and on the eigenvalues of A) such that for any integer $m \geq n + 1$, for any $0 \leq s_1 < \ldots < s_m \leq T_0$, the linear transformations $l_{s_1,\ldots,s_m} : \mathbb{R}^{dm} \to \mathbb{R}^n$,*

$$l_{s_1,\ldots,s_m}(y_1, \ldots, y_m) = \sum_{j=1}^{m} e^{s_j A} B y_j, \qquad \text{are onto.} \tag{2.5}$$

Proof. The proof is divided into two parts.

I Part. We define $T_0 > 0$.

Let (λ_j) be the distinct complex eigenvalues of A, $j = 1, \ldots, k$ (with $k \leq n$). Consider the following complex polynomial: $p(\lambda) = \prod_{j=1}^{k}(\lambda - \lambda_j)^n$, $\lambda \in \mathbb{C}$, and the corresponding ordinary linear differential operator $p(D)$ of order n,

$$p(D)y(t) = \big(\prod_{j=1}^{k}(D - \lambda_j)^n\big)y(t) = y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n, \quad t \in \mathbb{R},$$

where $y \in C^n(\mathbb{R})$, $a_i \in \mathbb{C}$, and $y^{(i)}$ denotes the i-derivative of y, $i = 1, \ldots, n$.

By a result due to Nehari (see [17]) we know, in particular, that there exists $T_0 > 0$ (depending on n and on the coefficients $a_1, \ldots, a_n$) such that *any* non-trivial solution $y(t)$ to the equation $p(D)y = 0$ has at most n zeros on $[-T_0, T_0]$. By this theorem, we deduce that the following quasi-polynomials

$$y(t) = \sum_{j=1}^{k}\sum_{r=0}^{n-1} c_{rj} e^{\lambda_j t} t^r, \tag{2.6}$$

which are solutions for $p(D)y = 0$ (see, for instance, [3, Chapter 3]), have always at most n zeros on $[-T_0, T_0]$ no matter what are the complex coefficients c_{rj} (except the trivial case in which all c_{rj} are zero).

II Part. We prove the assertion.

Introduce the following linear and bounded operators (depending on $t > 0$)

$$L_t : L^2([0,t]; \mathbb{R}^d) \to \mathbb{R}^n, \quad L_t u = \int_0^t e^{sA} B u(s) ds, \quad u \in L^2([0,t]; \mathbb{R}^d).$$

The controllability condition is equivalent to the fact that each L_t is onto, $t > 0$ (see, for instance, [27, Chapter 1]). Hence, in particular, $\mathrm{Im}(L_{T_0}) = \mathbb{R}^n$ ($T_0 > 0$ is defined in the first part of the proof). To prove the assertion it is enough to show that

$$\mathrm{Im}(L_{T_0}) \subset \mathrm{Im}\,(l_{s_1,\ldots,s_m}) \tag{2.7}$$

for any $0 \leq s_1 < \ldots < s_m \leq T_0$, and $m \geq n+1$. We fix $m \geq n+1$ and take $(s_1, \ldots, s_m)$ with $0 \leq s_1 < \ldots < s_m \leq T_0$. Let $v \in \mathbb{R}^n$, $v \neq 0$, be orthogonal to $\mathrm{Im}(l_{s_1,\ldots,s_m})$. Assertion (2.7) follows if we prove that

$$\langle v, L_{T_0} u\rangle = 0, \quad \text{for any } u \in L^2([0,T_0]; \mathbb{R}^d). \tag{2.8}$$

To this purpose, note that the orthogonality of v to $\mathrm{Im}(l_{s_1,\ldots,s_m})$ is equivalent to $B^* e^{s_j A^*} v = 0$, for $j = 1, \ldots, m$, i.e.,

$$\langle B^* e^{s_j A^*} v, e_k \rangle = 0, \quad j = 1, \ldots, m, \;\; k = 1, \ldots, d, \tag{2.9}$$

where (e_k) is the canonical basis in $\mathbb{R}^d$. Note that each mapping $s \mapsto \langle B^* e^{sA^*} v, e_k \rangle$, $k = 1, \ldots, d$, is a quasi-polynomial like (2.6). Since $m \geq n + 1$, condition (2.9) implies that each mapping $\langle B^* e^{sA^*} v, e_k \rangle$ is identically zero on $[0, T_0]$ by the first part of the proof.

It follows that

$$\langle L_{T_0} u, v \rangle = \int_0^{T_0} \langle u(s), B^* e^{sA^*} v \rangle ds = 0,$$

for any $u \in L^2([0, T_0]; \mathbb{R}^d)$. This implies that v is orthogonal to $\mathrm{Im}(L_{T_0})$ and so (2.7) holds. The proof is complete. ∎

Lemma 2.3. *Let $L : \mathbb{R}^p \to \mathbb{R}^q$, $p \geq q$, be an onto linear transformation. Let γ be a probability measure on $\mathbb{R}^p$ having a density h (with respect to the Lebesgue measure). Then the probability measure $L \circ \gamma$, image of γ under L, has a density on $\mathbb{R}^q$.*

Proof. Since the result is clear when $p = q$, let us assume that $p > q$. We identify L with a $q \times p$ matrix with respect to the canonical bases $(f_i)_{1 \leq i \leq p}$ in $\mathbb{R}^p$ and $(e_i)_{1 \leq i \leq q}$ in $\mathbb{R}^q$. Consider the transposed matrix L^* and complete the system of vectors $L^* e_1, \ldots, L^* e_q$ with vectors $f_{i_1}, \ldots f_{i_{p-q}}$ in order to get a basis in $\mathbb{R}^p$. Define an invertible $p \times p$ matrix S having the vectors $L^* e_1, \ldots, L^* e_q, f_{i_1}, \ldots f_{i_{p-q}}$ as rows. If $\pi : \mathbb{R}^p \to \mathbb{R}^q$ is the projection on the first q coordinates, we have that $L = \pi \circ S$. Indeed, for any $x \in \mathbb{R}^p$,

$$\pi(Sx) = (\langle e_1, Lx \rangle_{\mathbb{R}^q}, \ldots, \langle e_q, Lx \rangle_{\mathbb{R}^q}) = Lx.$$

Fix any Borel set $B \subset \mathbb{R}^q$. Using also the Fubini theorem, we get

$$\int_{\mathbb{R}^q} I_B(x)(L \circ \gamma)(dx) = \int_{\mathbb{R}^p} I_B(\pi(Sz)) h(z) dz = \frac{1}{|\det(S)|} \int_{\mathbb{R}^p} I_B(\pi(y)) h(S^{-1}(y)) dy$$

$$= \frac{1}{|\det(S)|} \int_B dy_1 \ldots dy_q \int_{\mathbb{R}^{p-q}} h \circ S^{-1}(y_1, \ldots, y_p) dy_{q+1} \ldots dy_p.$$

It follows that $L \circ \gamma$ has the density

$$(y_1, \dots, y_q) \mapsto \frac{1}{|\det(S)|} \int_{\mathbb{R}^{p-q}} h \circ S^{-1}(y_1, \dots, y_p) dy_{q+1} \dots dy_p.$$

■

Proof of Theorem 1.1. We will use Lemma 2.2 and adapt the method of the proof of [23, Theorem 27.7], based on [25] and [9]. Let $T_0 > 0$ be as in Lemma 2.2. Using Proposition 2.1 and the semigroup property of (P_t), in order to prove the assertion it is enough to show that the law of Y_t (see (2.1)) is absolutely continuous for any $t \in (0, T_0)$.

Recall that for an arbitrary Borel measure γ on $\mathbb{R}^n$, we have the unique measure decomposition

$$\gamma = \gamma_{ac} + \gamma_s \tag{2.10}$$

where γ_{ac} has a density and γ_s is singular with respect to the Lebesgue measure.

Define, for $N \in \mathbb{N}$ sufficiently large, say $N \geq N_0$ with $1/N_0 < r$, the measure ν_N having density $I_{\{1/N \leq |x| \leq r\}}$ with respect to ν, i.e.

$$\nu_N = \nu \, I_{\{1/N \leq |x| \leq r\}} \quad \text{and} \quad Z_t^N = \sum_{0<s\leq t,\, \frac{1}{N} \leq |\triangle Z_s| \leq r} \triangle Z_s, \quad t \geq 0,$$

(the measure ν_N has density $I_{\{1/N \leq |x| \leq r\}}$ with respect to the measure ν defined in (2.3)) and $\triangle Z_s = Z_s - Z_{s-}$ ($Z_{s-} = \lim_{h \to 0^-} Z_{s+h}$). The process (Z_t^N) is a compound Poisson process and its Lévy measure is just ν_N. By the hypotheses, for any $N \geq N_0$, ν_N *has a density.* Moreover since ν is infinite, we have that

$$c_N = \nu_N(\mathbb{R}^d) \to \infty \quad \text{as} \quad N \to \infty.$$

It is well known that (Z_t^N) and $(Z_t - Z_t^N)$ are independent Lévy processes (see, for instance, [1] or [21, Chapter 1]). It follows, in particular, that the random variables

$$Y_t^N = \int_0^t e^{(t-s)A} B dZ_s^N \quad \text{and} \quad Y_t - Y_t^N = \int_0^t e^{(t-s)A} B d(Z - Z^N)_s \ \text{ are independent,} \tag{2.11}$$

for any $N \geq N_0$, $t > 0$. Fix $t \in (0, T_0)$ and denote by μ *the law of the random variable* Y_t *and by* μ_N *the one of* Y_t^N.

Since $\mu = \mu_N * \beta_N$ (where β_N is the law of $Y_t - Y_t^N$), we have by (2.10)

$$\mu = (\mu_N)_{ac} * (\beta_N)_s + (\mu_N)_s * (\beta_N)_s + (\mu_N)_{ac} * (\beta_N)_{ac} + (\mu_N)_s * (\beta_N)_{ac}.$$

By [23, Lemma 27.1]) we deduce that $(\mu_N)_{ac} * (\beta_N)_s + (\mu_N)_{ac} * (\beta_N)_{ac} + (\mu_N)_s * (\beta_N)_{ac}$ is absolutely continuous and so $\mu_s = \big((\mu_N)_s * (\beta_N)_s\big)_s$ and

$$\mu_s(\mathbb{R}^n) \le (\mu_N)_s * (\beta_N)_s\big(\mathbb{R}^n\big) \le (\mu_N)_s(\mathbb{R}^n), \quad \text{for any } N \ge N_0. \tag{2.12}$$

Now we compute μ_N which coincides with the law of $\int_0^t e^{sA} B dZ_s^N$.

First, note that the law of Z_t^N is given by

$$e^{-c_N t}\delta_0 + e^{-c_N t} \sum_{k \ge 1} \frac{(c_N t)^k}{k!} (\tilde{\nu_N})^k, \text{ where } c_N = \nu_N(\mathbb{R}^d), \quad \tilde{\nu_N} = \frac{\nu_N}{c_N},$$

$(\tilde{\nu_N})^k = \tilde{\nu}_N * \ldots * \tilde{\nu}_N$ (k-times). Then consider a sequence (ξ_i) of independent random variables having the same exponential law of intensity c_N. Introduce another sequence (U_i) of independent random variables (independent also of (ξ_i)) having the same law $\tilde{\nu}_N$.

It is not difficult to check that the probability measure μ_N coincides with the law of the following random variable:

$$0 \cdot 1_{\{\xi_1 > t\}} + \sum_{k \ge 1} 1_{\{\xi_1 + \ldots + \xi_k \le t < \xi_1 + \ldots + \xi_{k+1}\}} \Big(e^{\xi_1 A} B U_1 + \ldots + e^{(\xi_1 + \ldots + \xi_k)A} B U_k \Big).$$

Note that the events $H_0 = \{\xi_1 > t\}$, $H_k = \{\xi_1 + \ldots + \xi_k \le t < \xi_1 + \ldots + \xi_{k+1}\}$ are all disjoint, $k \ge 1$. Since, for any $f \in B_b(\mathbb{R}^n)$,

$$f(\sum_{k \ge 0} X_k 1_{H_k}) = \sum_{k \ge 0} f(X_k) 1_{H_k},$$

where $X_0 = 0$ and $X_k = e^{\xi_1 A}BU_1 + \ldots + e^{(\xi_1+\ldots+\xi_k)A}BU_k$, $k \geq 1$, we get

$$\mathbb{E}f(Y_t^N) = e^{-c_N t}f(0) + R_N, \quad \text{where}$$

$$R_N = \mathbb{E}f\big(\sum_{k\geq 1} 1_{\{\xi_1+\ldots+\xi_k \leq t < \xi_1+\ldots+\xi_{k+1}\}}\big(e^{\xi_1 A}BU_1 + \ldots + e^{(\xi_1+\ldots+\xi_k)A}BU_k\big)\big)$$

$$= \sum_{k\geq 1} \mathbb{E}f\Big(1_{\{\xi_1+\ldots+\xi_k \leq t < \xi_1+\ldots+\xi_{k+1}\}}\big(e^{\xi_1 A}BU_1 + \ldots + e^{(\xi_1+\ldots+\xi_k)A}BU_k\big)\Big)$$

$$= \sum_{k=1}^{\infty} \int_{t_1+\ldots+t_k \leq t \leq t_1+\ldots+t_{k+1}} (c_N)^{k+1} e^{-c_N(t_1+\ldots+t_{k+1})} dt_1 \ldots dt_{k+1}$$

$$\cdot \int_{\mathbb{R}^{dk}} f(e^{t_1 A}By_1 + \ldots + e^{(t_1+\ldots+t_k)A}By_k)\tilde{\nu}_N(dy_1)\ldots\tilde{\nu}_N(dy_k)$$

$$= \sum_{k\geq 1} \int_{t_1+\ldots+t_k \leq t \leq t_1+\ldots+t_{k+1}} (c_N)^{k+1} e^{-c_N(t_1+\ldots+t_{k+1})} dt_1 \ldots dt_{k+1}\cdot$$

$$\cdot \int_{\mathbb{R}^n} f(y)\mu_{t_1,\ldots,t_k}(dy), \quad f \in B_b(\mathbb{R}^n),$$

where $\mu_{t_1,\ldots,t_k}$ is the probability measure on $\mathbb{R}^n$ which is the image of the product measure $\tilde{\nu}_N \times \ldots \times \tilde{\nu}_N$ (k-times) under the linear transformation $J_{t_1,\ldots,t_k}$ (independent of N) acting from $\mathbb{R}^{dk}$ into $\mathbb{R}^n$,

$$J_{t_1,\ldots,t_k}(y_1,\ldots,y_k) = e^{t_1 A}By_1 + \ldots + e^{(t_1+\ldots+t_k)A}By_k,$$

where $y_i \in \mathbb{R}^d$, $i = 1,\ldots,k$. For any $k \geq n+1$, $t_1 \geq 0$, $t_i > 0$, $i = 2,\ldots,k$, we have $0 \leq t_1 < \ldots < t_1 + \ldots + t_k \leq T_0$ and

$$J_{t_1,\ldots,t_k} = l_{t_1,\ldots,t_1+\ldots+t_k}$$

(see (2.5) and recall that $t \in (0,T_0)$). Applying Lemma 2.2, we obtain that, for any $k \geq n+1$, $t_i > 0$, $i = 1,\ldots,k$, the linear transformation $J_{t_1,\ldots,t_k}$ is *onto*. Therefore, by Lemma 2.3, the measure $\mu_{t_1,\ldots,t_k}$ has a density $g_{t_1,\ldots,t_k} \in L^1(\mathbb{R}^n)$, for any $k \geq n+1$, $t_i > 0$, $i = 1,\ldots,k$. Using this fact, we write

$$\mu_N = \mu_N^1 + \mu_N^2, \quad \text{where} \quad \mu_N^1 = e^{-c_N t}\delta_0 +$$

$$+ \sum_{k=1}^{n} \int_{t_1+\ldots+t_k < t < t_1+\ldots+t_{k+1}} (c_N)^{k+1} e^{-c_N(t_1+\ldots+t_{k+1})} \mu_{t_1,\ldots,t_k}\, dt_1 \ldots dt_{k+1},$$

and μ_N^2 has the following density on $\mathbb{R}^n$:

$$y \mapsto \sum_{k>n} \int_{t_1+\ldots+t_k < t < t_1+\ldots+t_{k+1}} (c_N)^{k+1} e^{-c_N(t_1+\ldots+t_{k+1})} g_{t_1,\ldots,t_k}(y) dt_1 \ldots dt_{k+1}.$$

Therefore

$$
\begin{aligned}
&(\mu_N)_s(\mathbb{R}^n) \le \mu_N^1(\mathbb{R}^n) \\
&= e^{-c_N t} + \sum_{k=1}^{n} \int_{t_1+\ldots+t_k<t<t_1+\ldots+t_{k+1}} (c_N)^{k+1} e^{-c_N(t_1+\ldots+t_{k+1})} dt_1 \ldots dt_{k+1} \longrightarrow 0,
\end{aligned}
$$

as $N \to \infty$, since $c_N \to \infty$ by hypothesis. By (2.12), we immediately get that $\mu_s = 0$. This gives the assertion. The proof is complete.

■

3 Proof of the C^∞-result

We pass now to the *proof of Theorem 1.3*. To obtain C^∞-regularity of the law at time t of the Ornstein-Uhlenbeck process (1.1) we will be estimating its characteristic function.

We fix $t > 0$. It is enough to show that the law μ_t of Y_t (see (2.1)) has a density $p_t \in L^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ with all bounded derivatives. To this purpose, note that by (2.2) the characteristic function of μ_t is

$$\hat{\mu}_t(y) = \exp\Big(-\int_0^t \psi(B^* e^{sA^*} y) ds\Big), \quad y \in \mathbb{R}^n.$$

We claim that there exist a_t and $c_t > 0$ such that, for any $y \in \mathbb{R}^n$, $|y| \ge 1$,

$$\big|\exp\Big(-\int_0^t \psi(B^* e^{sA^*} y) ds\Big)\big| \le c_t e^{-a_t |y|^\alpha}. \tag{3.1}$$

This will imply in particular that $\hat{\mu}_t \in L^1(\mathbb{R}^n)$. Then, by using the Fourier inversion formula (see [23, Propositions 2.5]) we will get the assertion.

It is not restrictive to assume that $Q = 0$ and $a = 0$ in (2.3), i.e., that (Z_t) has no Gaussian component. For any $y \in \mathbb{R}^n$, we have

$$\Big|\exp\Big(-\int_0^t \psi(B^* e^{sA^*} y) ds\Big)\Big| = \exp\Big(-\int_0^t ds \int_{\mathbb{R}^d} \big(1 - \cos(\langle B^* e^{sA^*} y, z\rangle)\big)\nu(dz)\Big).$$

First, note that condition (1.4) is equivalent to the fact that

$$\int_{\{z \in \mathbb{R}^d \,:\, |\langle z,k\rangle| \le 1\}} \langle z, k\rangle^2 \, \nu(dz) \ge C\, |k|^\alpha, \tag{3.2}$$

for sufficiently large $k \in \mathbb{R}^d$, say $|k| \geq c_0$. To see this, it is enough to change in the condition (1.4), the vector h to the vector k/r. Fix $y \in \mathbb{R}^n$ with $|y| \geq 1$; using also the inequality $1 - \cos(u) \geq c_1 |u|^2$, if $|u| \leq \pi$, we find

$$
\begin{aligned}
&\int_0^t ds \int_{\mathbb{R}^d} \big(1 - \cos(\langle B^* e^{sA^*} y, z\rangle)\big)\nu(dz) \\
&\geq c_1 \int_0^t ds \int_{\{z \in \mathbb{R}^d \,:\, |\langle B^* e^{sA^*} y, z\rangle| \leq 1\}} \langle B^* e^{sA^*} y, z\rangle^2 \, \nu(dz) \\
&\geq c_1 \int_0^t 1_{\{s \in [0,t] \,:\, |B^* e^{sA^*} y| \geq c_0\}} \, ds \int_{\{z \in \mathbb{R}^d \,:\, |\langle B^* e^{sA^*} y, z\rangle| \leq 1\}} \langle B^* e^{sA^*} y, z\rangle^2 \, \nu(dz) \\
&\geq c_1 C \int_0^t 1_{\{s \in [0,t] \,:\, |B^* e^{sA^*} y| \geq c_0\}} \, |B^* e^{sA^*} y|^\alpha ds.
\end{aligned}
$$

Set $M_t = \sup\{|B^* e^{sA^*} h| \;:\; s \in [0,t],\, |h| \leq 1,\, h \in \mathbb{R}^n\}$; since $\left|\frac{B^* e^{sA^*} y}{|y|\, M_t}\right| \leq 1$, $s \in [0,t]$, we get

$$
\begin{aligned}
&c_1 C \int_0^t 1_{\{s \in [0,t] \,:\, |B^* e^{sA^*} y| \geq c_0\}} \, |B^* e^{sA^*} y|^\alpha ds \\
&\geq c_1 C \, |y|^\alpha \, M_t^\alpha \int_0^t 1_{\{s \in [0,t] \,:\, |B^* e^{sA^*} y| \geq c_0\}} \left|\frac{B^* e^{sA^*} y}{|y|\, M_t}\right|^\alpha ds \\
&\geq \; c_1 C \, |y|^\alpha \, M_t^\alpha \int_0^t 1_{\{s \in [0,t] \,:\, |B^* e^{sA^*} y| \geq c_0\}} \left|\frac{B^* e^{sA^*} y}{|y|\, M_t}\right|^2 ds.
\end{aligned}
$$

Let us recall that the rank condition (1.2) is equivalent to the existence of $C_t > 0$ such that, for any $u \in \mathbb{R}^n$, $\int_0^t |B^* e^{sA^*} u|^2 \, ds \geq C_t |u|^2$ (see [27]). Moreover

$$
\int_0^t 1_{\{s \,:\, |B^* e^{sA^*} y| \leq c_0\}} \left|\frac{B^* e^{sA^*} y}{|y|\, M_t}\right|^2 ds \leq \frac{c_0^2 t}{|y|^2 \, M_t^2}
$$

This implies that, for any $y \in \mathbb{R}^n$, with $|y| \geq 1$,

$$
\begin{aligned}
&c_1 C \, |y|^\alpha \, M_t^\alpha \int_0^t 1_{\{s \,:\, |B^* e^{sA^*} y| \geq c_0\}} \left|\frac{B^* e^{sA^*} y}{|y|\, M_t}\right|^2 ds \\
&\geq c_1 C C_t \, |y|^\alpha \, M_t^{\alpha - 2} - c_1 C \, |y|^\alpha \, M_t^\alpha \int_0^t 1_{\{s \,:\, |B^* e^{sA^*} y| \leq c_0\}} \left|\frac{B^* e^{sA^*} y}{|y|\, M_t}\right|^2 ds \\
&\geq c_1 C C_t \, |y|^\alpha \, M_t^{\alpha - 2} - c_1 C \, |y|^\alpha \, M_t^\alpha \frac{c_0^2 t}{|y|^2 \, M_t^2}.
\end{aligned}
$$

We get, for any $y \in \mathbb{R}^n, |y| \geq 1$,

$$
\int_0^t ds \int_{\mathbb{R}^d} \big(1 - \cos(\langle B^* e^{sA^*} y, z\rangle)\big)\nu(dz) \geq c_1 C C_t M_t^{\alpha-2} \, |y|^\alpha - c_1 C c_0^2 t M_t^{\alpha - 2}.
$$

The assertion (3.1) is proved.

Finally, by the Fourier inversion formula,

$$p_t(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle y,h\rangle} \exp\Big(- \int_0^t \psi(B^* e^{sA^*} h) ds\Big) dh, \;\; y \in \mathbb{R}^n, \tag{3.3}$$

is the density of μ_t. Differentiating under the integral sign, we get easily the assertion. The proof is complete. ■

Remark 3.1. It follows from Theorem 1.3 that for any Borel function f *with compact support* one has $P_t f \in C_b^\infty(\mathbb{R}^n)$, for any $t > 0$ (i.e., $P_t f \in C^\infty(\mathbb{R}^n)$ with all bounded derivatives of any order) where $P_t f(x) = \int_{\mathbb{R}^n} f(z) p_t(z - e^{tA} x) dz$. We do not know if this regularizing effect holds for all $f \in B_b(\mathbb{R}^n)$ as we are unable to show that for a given multi-index β the partial derivative $D^\beta p_t$ is integrable on $\mathbb{R}^n$.

Acknowledgment The authors thank the referee for the careful reading of the original manuscript, and for giving useful comments.

References

[1] D. APPLEBAUM, *Lévy processes and stochastic calculus,* Cambridge Studies in Advanced Mathematics 93 (Cambridge University Press, 2004).

[2] O. E. BARNDORFF-NIELSEN and N. SHEPARD, 'Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics', *J. R. Stat. Soc. Ser. B Stat. Methodol.* 63 (2001) 167-241.

[3] G. BIRKHOFF, G. ROTA, *Ordinary differential equations,* fourth edition (John Wiley & Sons, 1989).

[4] J. M. BISMUT, 'Calcul des variations stochastique et processus de sauts' (French) [Stochastic calculus of variations and jump processes], *Z. Wahrsch. Verw. Gebiete* 63 (1983) 147-235.

[5] P. J. BROCKWELL, 'Lévy-driven CARMA processes', *Ann. Inst. Statist. Math.* 53 (2001) 113-124.

[6] R. CONT and P. TANKOV, *Financial modelling with jump processes* (Chapman & Hall/CRC, Boca Raton, FL, 2004).

[7] G. DA PRATO and J. ZABCZYK, *Stochastic Equations in Infinite Dimensions* (Cambridge University Press, 1992).

[8] G. DA PRATO and J. ZABCZYK, *Second Order Partial Differential Equations in Hilbert Spaces,* London Mathematical Society Lecture Note Series 293 (Cambridge University Press, 2002).

[9] M. FISZ and V. S. VARADARAJAN, 'A condition for absolute continuity of infinitely divisible distribution functions', *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 1 (1962/1963) 335-339.

[10] P. GARBACZEWSKI and R. OLKIEWICZ, 'Ornstein-Uhlenbeck-Cauchy process', *Journal of Mathematical Physics* 41 (2000), 6843-6860.

[11] L. HÖRMANDER, 'Hypoelliptic second order differential operators', *Acta Math.* 119 (1967) 147-171.

[12] J. HAWKES, 'Potential theory of Lévy processes', *Proc. London Math. Soc. (3)* 38 (1979) 335-352.

[13] Y. ISHIKAWA, H. KUNITA, 'Malliavin calculus on the Wiener-Poisson space and application to canonical SDE with jumps', *Stoch. Proc. Applic.* 116 (2006) 1743-1769.

[14] A. N. KOLMOGOROV, 'Zufällige Bewegungen', *Ann. of Math. (2)* 116 (1934) 116-117.

[15] T. KOMATSU and A. TAKEUCHI, 'On the smoothness of PDF of solutions to SDE of jump type', *Int. J. Differ. Equ. Appl.* 2 (2001) 141-197.

[16] A. NEGORO and M. TSUCHIYA, 'Stochastic processes and semigroups associated with degenerate Lévy generating operators', *Stochastics Stochastics Rep.* 26 (1989) 29-61.

[17] Z. NEHARI, 'On the zeros of solutions of n-th order linear differential equations', *J. London Math. Soc.* 39 (1964) 327-332.

[18] J. PICARD, 'On the existence of smooth densities for jump processes', *Probab. Theory Related Fields* 105 (1996) 481-511.

[19] E. PRIOLA and J. ZABCZYK, 'Liouvile theorems for non-local operators', J. Funct. Anal. 216 (2004), 455-490.

[20] E. PRIOLA and J. ZABCZYK, 'Harmonic functions for generalized Mehler semigroups', *Stochastic partial differential equations and applications-VII* (eds G. DA PRATO and L. TUBARO), Lect. Notes Pure Appl. Math. 245 (Chapman & Hall/CRC, 2006), 243-256.

[21] P. E. PROTTER, *Stochastic integration and differential equations. Second edition,* Applications of Mathematics 21 (Springer, Berlin, 2004).

[22] M. RÖCKNER and F. Y. WANG, 'Harnack and Functional Inequalities for Generalised Mehler Semigroups', *J. Funct. Anal.* 203 (2003) 237-261.

[23] K. I. SATO, *Lévy processes and infinite divisible distributions* (Cambridge University Press, 1999).

[24] K.I. SATO, T. WATANABE, K. YAMAMURO and M. YAMAZATO, 'Multidimensional process of Ornstein-Uhlenbeck type with nondiagonalizable matrix in linear drift terms', *Nagoya Math. J.* 141 (1996) 45-78.

[25] H. G. TUCKER, 'Absolute continuity of infinitely divisible distributions', *Pacific J. Math.* 12 (1962) 1125-1129.

[26] J. ZABCZYK, 'Stationary Distribution for linear equations driven by general noise', *Bull. Acad. Po. Sci.* 31 (1983) 197-209.

[27] J. ZABCZYK, *Mathematical Control Theory: An introduction* (Birkhauser, 1992).