

On linear evolution equations with cylindrical Lévy noise

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Abstract: We study an infinite-dimensional Ornstein-Uhlenbeck process (X_t) in a given Hilbert space H . This is driven by a cylindrical symmetric Lévy process without a Gaussian component and taking values in a Hilbert space U which usually contains H . We give if and only if conditions under which X_t takes values in H for some $t > 0$ or for all $t > 0$. Moreover, we prove irreducibility for (X_t) .

1 Introduction and notation

There is an increasing interest in stochastic evolution equations driven by Lévy noise. We refer to the recent monograph [10] which also discusses several applications.

In this note we concentrate on the linear stochastic differential equation

$$\begin{cases} dX_t = AX_t dt + dZ_t, & t \geq 0, \\ X_0 = x \in H, \end{cases} \quad (1.1)$$

in a real separable Hilbert space H driven by an infinite dimensional cylindrical symmetric Lévy process $Z = (Z_t)$. The process Z may take values in a Hilbert space

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U usually greater than H . Moreover we assume that A is a linear possibly unbounded operator which generates a C_0 -semigroup (e^{tA}) on H .

Solutions of (1.1), called (generalised) Ornstein-Uhlenbeck processes, have recently received a lot of attention (see, for instance, [3], [1], [7], [6], [9], [11], [10], [14] and [2]). Transition semigroups determined by solutions $X = (X_t^x)$ to (1.1) are also studied under the name of generalized Mehler semigroups.

In the case when Z is a cylindrical Wiener process the theory of equations (1.1) is well understood (see [4], [5] and the references therein). The situation changes completely in the Lévy noise case and new phenomena appear. For instance, the càdlàg property of trajectories in H can be proved only in very special cases (see Remark 2.10) and is an open question in general. Note that in [7] it is proved that trajectories of (X_t^x) are càdlàg only in some enlarged Hilbert space containing H .

In this note we consider cylindrical Lévy process $Z = (Z_t)$ defined by the orthogonal expansion

$$Z_t = \sum_{n \geq 1} \beta_n Z_t^n e_n, \quad t \geq 0, \quad (1.2)$$

where (e_n) is an orthonormal basis of H . We also assume

Hypothesis 1.1. $Z^n = (Z_t^n)$ are independent, real valued, symmetric, identically distributed Lévy processes without a Gaussian part defined on a fixed stochastic basis. Moreover, (β_n) is a given (possibly unbounded) sequence of positive real numbers.

In our previous paper [14], we have considered the case in which (Z_t^n) are independent, real valued, normalized, symmetric α -stable processes, $\alpha \in (0, 2)$. For the linear equation (1.1) in [14] we have proved p -integrability of trajectories in H , $p \in (0, \alpha)$, and characterized the support of (X_t^x, X_T^x) in $L^p(0, T; H) \times H$. Moreover, we have established the strong Feller property for the transition Markov semigroup associated to (1.1). We are not able to prove such results in the present more general situation.

This note can be considered as a preliminary step towards an extension of [14] to general Lévy processes. In fact in Theorem 2.8 we provide if and only if conditions under which (X_t^x) takes values in H . It turns out that if there exists a positive time t_0 such that $X_{t_0}^x \in H$, \mathbb{P} -a.s., then for all $t > 0$, we have that $X_t^x \in H$, \mathbb{P} -a.s.. In Proposition 2.11 we consider a class of symmetric one dimensional Lévy processes Z_t^n , which includes the α -stable processes, and which satisfies the conditions of Theorem 2.8. For such processes we also show existence and uniqueness of invariant measure. The Markov property and irreducibility are proved in Theorems 2.8 and 3.3.

The results of the paper apply in particular to stochastic heat equations with Dirichlet boundary conditions (see Example 2.12).

Let us mention that in the recent paper [2] a different cylindrical Lévy noise Z is studied by subordinating a cylindrical Wiener process, given by (1.2) with (Z_t^n) independent real valued Wiener processes. It is difficult to judge at the moment which class of cylindrical Lévy noises will suit better modelling purposes.

As far as the strong Feller property for (1.1) is concerned we stress two different difficulties. One difficulty is related to the fact that very rarely for non-Gaussian infinitely divisible measures in Hilbert spaces formulae for the Radon-Nikodym derivatives are known. Another problem is that the well-known Bismut-Elworthy-Li formula is not available in the non-Gaussian case. A related formula, but requiring a non trivial Gaussian component in the Lévy noise, was established in finite dimensions in [13] and generalized to infinite dimensions in [15].

The space H will denote a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. We will fix an orthonormal basis (e_n) in H . Through the basis (e_n) we will often identify H with l^2 . More generally, for a given sequence $\rho = (\rho_n)$ of real numbers, we set

$$l_\rho^2 = \{(x_n) \in \mathbb{R}^\mathbb{N} : \sum_{n \geq 1} x_n^2 \rho_n^2 < +\infty\}. \quad (1.3)$$

The space l_ρ^2 becomes a separable Hilbert space with the inner product: $\langle x, y \rangle = \sum_{n \geq 1} x_n y_n \rho_n^2$, for $x = (x_n), y = (y_n) \in l_\rho^2$.

Let us recall that a Lévy process (Z_t) with values in H is an H -valued process defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, having stationary independent increments, càdlàg trajectories, and such that $Z_0 = 0$, \mathbb{P} -a.s.. One has that

$$\mathbb{E}[e^{i\langle Z_t, s \rangle}] = \exp(-t\psi(s)), \quad s \in H, \quad (1.4)$$

where the exponent ψ can be expressed by the following infinite dimensional Lévy-Khintchine formula,

$$\psi(s) = \frac{1}{2} \langle Qs, s \rangle - i \langle a, s \rangle - \int_H \left(e^{i\langle s, y \rangle} - 1 - \frac{i\langle s, y \rangle}{1 + |y|^2} \right) \nu(dy), \quad s \in H. \quad (1.5)$$

Here Q is a symmetric non-negative trace class operator on H , $a \in H$ and ν is the Lévy measure or the jump intensity measure associated to (Z_t) , i.e., ν is a σ -finite Borel measure on H such that $\nu(\{0\}) = 0$ and $\int_H (|y|^2 \wedge 1) \nu(dy) < +\infty$ (see [16] and [10]).

According to Proposition 2.4 our cylindrical Lévy process Z appearing in (1.1) is a Lévy process taking values in the Hilbert space $U = l_\rho^2$, with a properly chosen weight ρ (see Remark 2.7).

2 The main result

Concerning equation (1.1), we make the following assumption.

Hypothesis 2.1. *$A : D(A) \subset H \rightarrow H$ is a self-adjoint operator such that the fixed basis (e_n) of H verifies: $(e_n) \subset D(A)$, $Ae_n = -\gamma_n e_n$ with $\gamma_n > 0$, for any $n \geq 1$, and $\gamma_n \rightarrow +\infty$.*

Clearly, under (i), $D(A) = \{x = (x_n) \in H : \sum_{n \geq 1} x_n^2 \gamma_n^2 < +\infty\}$. In addition A generates a compact C_0 -semigroup (e^{tA}) on H such that

$$e^{tA} e_k = e^{-\gamma_k t} e_k, \quad k \in \mathbb{N}, \quad t \geq 0.$$

Hypothesis 2.1 is also considered in [14] when (Z_t^n) are symmetric α -stable Lévy processes, $\alpha \in (0, 2)$.

Recall that we are assuming that (Z_t^n) are defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual assumptions.

Since the law of Z_t^n is symmetric, we have, for any $n \geq 1, t \geq 0$,

$$\mathbb{E}[e^{ihZ_t^n}] = e^{-t\psi(h)}, \quad h \in \mathbb{R}, \quad (2.1)$$

where

$$\psi(h) = \int_{\mathbb{R}} (1 - \cos(hy)) \nu(dy), \quad h \in \mathbb{R}, \quad (2.2)$$

and the Lévy measure ν is symmetric (i.e., $\nu(A) = \nu(-A)$, for any Borel set $A \subset \mathbb{R}$). This follows by the next elementary result.

Proposition 2.2. *A one dimensional Lévy process $L = (L_t)$ without Gaussian part has symmetric distribution at some time $t > 0$ if and only if its Lévy measure ν is symmetric. Moreover, if ν is symmetric then L_t has symmetric distribution at any time $t \geq 0$.*

Proof. Since (L_t) has no Gaussian part, according to the Lévy-Khintchine formula, we have $\mathbb{E}[e^{ihL_t}] = e^{-t\psi(h)}$, $h \in \mathbb{R}$, $t \geq 0$, with

$$\psi(h) = -iah - \int_{\mathbb{R}} \left(e^{ihy} - 1 - \frac{ihy}{1+y^2} \right) \nu(dy), \quad h \in \mathbb{R},$$

for some $a \in \mathbb{R}$. Define the reflection measure $\tilde{\nu}$ of ν , i.e., $\tilde{\nu}(A) = \nu(-A)$, for any Borel set $A \subset \mathbb{R}$. It is easy to check that also $\tilde{\nu}$ is a Lévy measure and moreover

$$\psi(-h) = iah - \int_{\mathbb{R}} \left(e^{ihy} - 1 - \frac{ihy}{1+y^2} \right) \tilde{\nu}(dy), \quad h \in \mathbb{R}.$$

Since L_t has symmetric distribution, we must have $\psi(h) = \psi(-h)$, $h \in \mathbb{R}$. By uniqueness of the Lévy-Khintchine formula (see [16, Theorem 8.1]) we obtain that $(-a, \tilde{\nu}) = (a, \nu)$. It follows that $a = 0$ and $\nu = \tilde{\nu}$. Therefore ν is symmetric. \square

We need the following lemma.

Lemma 2.3. *Let us consider a sequence of independent, symmetric, infinitely divisible real random variables ξ_n defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\mathbb{E}[e^{ih\xi_n}] = \exp \left[- \int_{\mathbb{R}} (1 - \cos(hy)) \nu_n(dy) \right], \quad h \in \mathbb{R}, \quad n \geq 1,$$

where ν_n are Lévy measures. The following assertions are equivalent.

- (i) $\sum_{n \geq 1} \xi_n^2 < +\infty$, \mathbb{P} -a.s.;
- (ii) $\sum_{n \geq 1} \int_{\mathbb{R}} (1 \wedge y^2) \nu_n(dy) < +\infty$.

Proof. We will use the following theorem (see, for instance [8], page 70-71): let U_n be a sequence of independent and symmetric real random variables; then the following statements are equivalent: $\sum_{n \geq 1} U_n$ converges in distribution; $\sum_{n \geq 1} U_n$ converges \mathbb{P} -a.s.; $\sum_{n \geq 1} U_n^2$ converges \mathbb{P} -a.s..

By the previous result, assertion (i) is equivalent to convergence in distribution of the sequence of random variables $(\sum_{n=1}^N \xi_n)$.

We have, for any $N \in \mathbb{N}$, $h \in \mathbb{R}$, using independence,

$$\mathbb{E}[e^{ih \sum_{n=1}^N \xi_n}] = \prod_{n=1}^N \mathbb{E}[e^{ih\xi_n}] = \prod_{n=1}^N e^{-\int_{\mathbb{R}} (1 - \cos(hy)) \nu_n(dy)}$$

$$= e^{-\sum_{n=1}^N \int_{\mathbb{R}} (1 - \cos(hy)) \nu_n(dy)} = e^{-\int_{\mathbb{R}} (1 - \cos(hy)) (\sum_{n=1}^N \nu_n)(dy)}$$

(i) \Rightarrow (ii). We are assuming convergence in distribution of the sequence $(\sum_{n=1}^N \xi_n)$. By [16, Theorem 8.7] the limiting distribution μ is again symmetric and infinitely divisible; the characteristic function of μ is given by

$$\exp\left(-\frac{1}{2}qh^2 - \int_{\mathbb{R}} (1 - \cos(hy)) \tilde{\nu}(dy)\right), \quad h \in \mathbb{R},$$

where

$$q \geq 0 \text{ and } \int_{\mathbb{R}} (y^2 \wedge 1) \tilde{\nu}(dy) < +\infty. \quad (2.3)$$

Moreover, for arbitrary bounded continuous functions f from \mathbb{R} into \mathbb{R} , vanishing on a neighborhood of 0, we have

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} f(y) \left(\sum_{n=1}^N \nu_n\right)(dy) = \int_{\mathbb{R}} f(y) \tilde{\nu}(dy). \quad (2.4)$$

If f in (2.4) is in addition a non-negative function, then $\int_{\mathbb{R}} f(y) (\sum_{n=1}^N \nu_n)(dy)$ is an increasing sequence. Thus, for any $N \in \mathbb{N}$, $f : \mathbb{R} \rightarrow \mathbb{R}_+$ continuous, bounded and vanishing on a neighborhood of 0,

$$\int_{\mathbb{R}} f(y) \left(\sum_{n=1}^N \nu_n\right)(dy) \leq \int_{\mathbb{R}} f(y) \tilde{\nu}(dy).$$

Since the function $y \mapsto y^2 \wedge 1$ is a pointwise limit of a monotone increasing sequence of non-negative functions f_k which are continuous, bounded and vanishing on a neighborhood of 0, we have, for any $k \geq 1$, $N \geq 1$,

$$\int_{\mathbb{R}} f_k(y) \left(\sum_{n=1}^N \nu_n\right)(dy) \leq \int_{\mathbb{R}} f_k(y) \tilde{\nu}(dy) \leq \int_{\mathbb{R}} (y^2 \wedge 1) \tilde{\nu}(dy).$$

Passing to the limit, as $k \rightarrow \infty$, in the left hand-side of the previous formula, we get assertion (ii).

(ii) \Rightarrow (i). By using the inequality

$$\int_{\mathbb{R}} (1 - \cos(hy)) \nu_n(dy) \leq \int_{\mathbb{R}} (1 \wedge (hy)^2) \nu_n(dy), \quad h \in \mathbb{R}, \quad n \geq 1,$$

we obtain that condition (ii) implies that the series

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} (1 - \cos(hy)) \nu_n(dy)$$

converges uniformly in h on compact sets of \mathbb{R} . By the Lévy convergence theorem, this gives convergence in distribution of the sequence $(\sum_{n=1}^N \xi_n)$ and concludes the proof. \square

Applying the previous result we can clarify when our cylindrical Lévy process $Z = (Z_t)$ takes values in H .

Proposition 2.4. *The following conditions are equivalent.*

- (i) $\sum_{n \geq 1} (\beta_n Z_{t_0}^n)^2 < +\infty$, \mathbb{P} -a.s., for some $t_0 > 0$;
- (ii) $\sum_{n \geq 1} (\beta_n Z_t^n)^2 < +\infty$, \mathbb{P} -a.s., for any $t \geq 0$;
- (iii) $\sum_{n \geq 1} \left(\beta_n^2 \int_{|y| < 1/\beta_n} y^2 \nu(dy) + \int_{|y| \geq 1/\beta_n} \nu(dy) \right) < +\infty$.

Proof. We first show that (i) implies (iii). Assertion (i) is equivalent to convergence in distribution of the sequence of random variables $(\sum_{n=1}^N \beta_n Z_{t_0}^n)$ (see the result mentioned at the beginning of the proof of Lemma 2.3). We have, for any $n \geq 1$, $h \in \mathbb{R}$,

$$\mathbb{E}[e^{ih\beta_n Z_{t_0}^n}] = e^{-t_0 \int_{\mathbb{R}} (1 - \cos(h\beta_n y)) \nu(dy)} = e^{-t_0 \int_{\mathbb{R}} (1 - \cos(hy)) \nu_n(dy)},$$

where ν_n is the image law of ν by the transformation: $y \mapsto \beta_n y$. Setting $\xi_n = \beta_n Z_{t_0}^n$, by Lemma 2.3 assertion (i) is equivalent to

$$\sum_{n \geq 1} \int_{\mathbb{R}} (1 \wedge y^2) \nu_n(dy) < +\infty.$$

Now assertion (iii) follows since, for any $n \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathbb{R}} (y^2 \wedge 1) \nu_n(dy) &= \int_{|y| < 1} y^2 \nu_n(dy) + \int_{|y| \geq 1} \nu_n(dy) \\ &= \int_{|\beta_n y| < 1} \beta_n^2 y^2 \nu(dy) + \int_{|\beta_n y| \geq 1} \nu(dy). \end{aligned}$$

Using again Lemma 2.3 we get that (iii) implies (ii) as well. The proof is complete. \square

Remark 2.5. Theorem 4.13 in [10] states that if condition (ii) in Proposition 2.4 holds then also (iii) is satisfied. However such theorem does not require symmetry of the Lévy process Z .

Remark 2.6. If (Z_t^n) are symmetric α -stable processes, $\alpha \in (0, 2)$, then $\nu(dy) = \frac{1}{|y|^{1+\alpha}} dy$ and so (ii) of Proposition 2.4 is equivalent to

$$\sum_{n \geq 1} \beta_n^\alpha < +\infty$$

as in [14].

Remark 2.7. Using Proposition 2.4, one gets that our cylindrical Lévy process Z is a Lévy process with values in the space l_ρ^2 , see (1.3), where (ρ_n) is a sequence of positive numbers such that

$$\sum_{n \geq 1} \left((\rho_n \beta_n)^2 \int_{|\rho_n \beta_n y| < 1} y^2 \nu(dy) + \int_{|\rho_n \beta_n y| \geq 1} \nu(dy) \right) < +\infty.$$

Let us come back to the Ornstein-Uhlenbeck process. According to Hypothesis 2.1, we may consider our equation (1.1) as an infinite sequence of independent one dimensional stochastic equations, i.e.,

$$dX_t^n = -\gamma_n X_t^n dt + \beta_n dZ_t^n, \quad X_0^n = x_n, \quad n \in \mathbb{N}, \quad (2.5)$$

with $x = (x_n) \in l^2 = H$. The *solution* is a stochastic process $X = (X_t^x)$ which takes values in $\mathbb{R}^{\mathbb{N}}$ with components

$$X_t^n = e^{-\gamma_n t} x_n + \int_0^t e^{-\gamma_n(t-s)} \beta_n dZ_s^n, \quad n \in \mathbb{N}, \quad t \geq 0. \quad (2.6)$$

Theorem 2.8. *Assume Hypotheses 1.1 and 2.1 and consider the process $X = (X_t^x)$ given in (2.6), $x \in H$. Define $\psi_0(u) = \int_{\{|y| \leq u\}} y^2 \nu(dy)$ and $\psi_1(u) = \int_{\{|y| > u\}} \nu(dy)$.*

The following assertions are equivalent.

- (i) $X_{t_0}^x \in H$, \mathbb{P} -a.s., for some $t_0 > 0$;
- (ii) $X_t^x \in H$, \mathbb{P} -a.s., for any $t \geq 0$;
- (iii) $\sum_{n \geq 1} \frac{1}{\gamma_n} \int_{\frac{1}{\beta_n}}^{\frac{1}{\beta_n} e^{\gamma_n t_0}} \left(\frac{1}{u^3} \psi_0(u) + \frac{1}{u} \psi_1(u) \right) du < +\infty$.

Moreover, under one of the previous assertions, we have

$$X_t^x = \sum_{n \geq 1} X_t^n e_n = e^{tA} x + Z_A(t), \quad \text{where} \quad (2.7)$$

$$Z_A(t) = \int_0^t e^{(t-s)A} dZ_s = \sum_{n \geq 1} \left(\int_0^t e^{-\gamma_n(t-s)} \beta_n dZ_s^n \right) e_n,$$

and the process (X_t^x) is \mathcal{F}_t -adapted and Markovian.

Proof. I step. We show that (i) is equivalent to the following condition

$$\sum_{n \geq 1} \frac{1}{\gamma_n} \int_{\frac{1}{\beta_n}}^{\frac{1}{\beta_n} e^{\gamma_n t_0}} \left(\frac{1}{u^3} \psi_0(u) + \frac{1}{u} \psi_1(u) \right) du < +\infty. \quad (2.8)$$

Let us consider the stochastic convolution

$$Y_t^n = Z_A^n(t) = \int_0^t e^{-\gamma_n(t-s)} \beta_n dZ_s^n, \quad n \in \mathbb{N}, \quad t \geq 0, \quad (2.9)$$

where the stochastic integral is a limit in probability of Riemann sums. We have, for any $h \in \mathbb{R}$, see (2.1),

$$\mathbb{E}[e^{ihY_{t_0}^n}] = \exp \left[- \int_0^{t_0} \psi(e^{-\gamma_n s} \beta_n h) ds \right] \quad (2.10)$$

where ψ is given in (2.1). By the Fubini theorem

$$\int_0^{t_0} \psi(e^{-\gamma_n s} \beta_n h) ds = \int_0^{t_0} ds \int_{\mathbb{R}} (1 - \cos(e^{-\gamma_n s} \beta_n h y)) \nu(dy)$$

$$= \int_0^{t_0} ds \int_{\mathbb{R}} (1 - \cos(hy)) \nu_{ns}(dy) = \int_{\mathbb{R}} (1 - \cos(hy)) \tilde{\nu}_n(dy),$$

where ν_{ns} is the image law of ν by the transformation: $y \mapsto \beta_n e^{-\gamma_n s} y$ and we have set

$$\tilde{\nu}_n(B) = \left(\int_0^{t_0} \nu_{ns} ds \right)(B) = \int_0^{t_0} \left(\int_{\mathbb{R}} I_B(\beta_n e^{-\gamma_n s} y) \nu(dy) \right) ds, \quad (2.11)$$

for any Borel set $B \subset \mathbb{R}$ (I_B is the indicator function of B). Setting $\xi_n = Y_{t_0}^n$, by Lemma 2.3, assertion (i) is equivalent to

$$\sum_{n \geq 1} \int_{\mathbb{R}} (1 \wedge y^2) \tilde{\nu}_n(dy) = \sum_{n \geq 1} \int_0^{t_0} \left(\int_{\mathbb{R}} (1 \wedge y^2) \nu_{ns}(dy) \right) ds < +\infty. \quad (2.12)$$

Let us fix $n \geq 1$. We have

$$\begin{aligned} \int_0^{t_0} \left(\int_{\mathbb{R}} (1 \wedge y^2) \nu_{ns}(dy) \right) ds &= \int_0^{t_0} \left(\int_{\mathbb{R}} (1 \wedge (\beta_n e^{-\gamma_n s} y)^2) \nu(dy) \right) ds \\ &= \int_0^{t_0} \left(\beta_n^2 e^{-2\gamma_n s} \int_{|y| \leq \frac{e^{\gamma_n s}}{\beta_n}} y^2 \nu(dy) + \int_{|y| > \frac{e^{\gamma_n s}}{\beta_n}} \nu(dy) \right) ds \\ &= \frac{1}{\gamma_n} \int_{\frac{1}{\beta_n}}^{\frac{e^{\gamma_n t_0}}{\beta_n}} \left(\frac{1}{u^3} \int_{|y| \leq u} y^2 \nu(dy) + \frac{1}{u} \int_{|y| > u} \nu(dy) \right) du. \end{aligned}$$

This shows that (2.12) is exactly (2.8).

II step. In order to prove equivalence between (i), (ii) and (iii) it remains to show that (i) implies (ii).

Note that if (2.8) holds for $t_0 > 0$, then it is also satisfied for any $0 \leq s \leq t_0$. Therefore, assertion (i) implies that

$$X_s^x \in H, \quad \mathbb{P} - a.s., \quad \text{for any } s \in [0, t_0]. \quad (2.13)$$

and so $Z_A(s) \in H$, $\mathbb{P} - a.s.$, for any $s \in [0, t_0]$.

We have the following identity on the product space $\mathbb{R}^{\mathbb{N}}$, \mathbb{P} -a.s.,

$$Z_A(T+h) - e^{hA} Z_A(T) = \int_T^{T+h} e^{(T+h-s)A} dZ_s = \int_0^h e^{(h-u)A} dZ_u^T, \quad (2.14)$$

for any $T, h \geq 0$, where $Z_u^T = Z_{T+u} - Z_T$, $u \geq 0$, is still a Lévy process with values in $\mathbb{R}^{\mathbb{N}}$. Note that

$$\int_0^h e^{(h-u)A} dZ_u^T$$

has the same law of $Z_A(h)$.

Combining (2.13) and identity (2.14) with $T = t_0$ and $h \in [0, t_0]$, we deduce that $\mathbb{P}(Z_A(r) \in H) = 1$, for any $r \in [t_0, 2t_0]$. By an iteration procedure, we infer that $\mathbb{P}(Z_A(r) \in H) = 1$, for any $r \geq 0$. This immediately implies condition (ii). The first part of the proof is finished.

III step. The property that (X_t^x) is \mathcal{F}_t -adapted is equivalent to the fact that each real process $\langle X_t^x, e_k \rangle$ is \mathcal{F}_t -adapted, for any $k \geq 1$, and this clearly holds.

The Markov property follows easily from the identity (2.14). \square

Remark 2.9. Note that

$$\lim_{\gamma \rightarrow 0} \frac{1}{\gamma} \int_{\frac{1}{\beta_n}}^{\frac{1}{\beta_n} e^\gamma} \left(\frac{1}{u^3} \psi_0(u) + \frac{1}{u} \psi_1(u) \right) du = \beta_n^2 \psi_0\left(\frac{1}{\beta_n}\right) + \psi_1\left(\frac{1}{\beta_n}\right).$$

This shows that Proposition 2.4 is a “limiting case” of Theorem 2.8, obtained when $\gamma_n = 0$, for any $n \geq 1$.

Remark 2.10. If the cylindrical Lévy process Z takes values in the Hilbert space H , i.e., if condition (ii) of Proposition 2.4 holds, then, by the Kotelenetz regularity result (see [10, Theorem 9.20]) trajectories of the process X which solves (1.1) are càdlàg with values in H . However such condition (ii) is a very restrictive assumption (see also Remark 2.6). We conjecture that the càdlàg property holds under much weaker conditions but, at the moment, this is an open problem.

In the next result we provide an application of Theorem 2.8 to Ornstein-Uhlenbeck processes driven by a quite general class of symmetric cylindrical Lévy noises (this class in particular includes the α -stable cylindrical processes).

Proposition 2.11. *Assume Hypotheses 1.1 and 2.1. Moreover, assume that (β_n) is a bounded sequence and that the symmetric Lévy measure ν appearing in (2.2) satisfies*

$$\int_1^{+\infty} \log(y) \nu(dy) < +\infty. \quad (2.15)$$

Finally, assume that

$$\sum_{n \geq 1} \frac{1}{\gamma_n} < +\infty.$$

Then the Ornstein-Uhlenbeck process $X = (X_t^x)$ given in (2.6) verifies assertions (i)-(iii) of Theorem 2.8. Moreover, X has an unique invariant measure.

Proof. First remark that (ii) of Theorem 2.8 is equivalent to

$$\sum_{n \geq 1} \beta_n^2 \left(\int_0^t e^{-\gamma_n(t-s)} dZ_s^n \right)^2 < +\infty, \quad \mathbb{P} - a.s., \quad t \geq 0.$$

Therefore, it is enough to check the result assuming that $\beta_n = 1$, for any $n \geq 1$.

We will check condition (iii), i.e.,

$$\sum_{n \geq 1} \frac{1}{\gamma_n} \int_1^{e^{\gamma_n}} \left(\frac{1}{u^3} \psi_0(u) + \frac{1}{u} \psi_1(u) \right) du < +\infty. \quad (2.16)$$

Let us fix $0 < 1 < b$. We first estimate the function f_0 ,

$$f_0(b) = \int_1^b \left(\frac{1}{u^3} \psi_0(u) + \frac{1}{u} \psi_1(u) \right) du.$$

We have, by using symmetricity and Fubini theorem,

$$\int_1^b \frac{1}{u} \psi_1(u) du = 2 \int_1^b \frac{1}{u} \left(\int_{y>u} \nu(dy) \right) du$$

$$\begin{aligned}
&= 2 \int_1^b \frac{1}{u} du \int_0^{+\infty} 1_{(u,+\infty)}(y) \nu(dy) = 2 \int_0^{+\infty} \left(\int_1^b \frac{1}{u} 1_{(u,+\infty)}(y) du \right) \nu(dy) \\
&= 2 \int_0^{+\infty} \left(\int_{1 \wedge y}^{b \wedge y} \frac{du}{u} \right) \nu(dy) = 2 \int_0^{+\infty} (\log(b \wedge y) - \log(1 \wedge y)) \nu(dy) \\
&= 2 \int_1^b \log(y) \nu(dy) + 2 \log(b) \nu((b, +\infty))
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\int_1^b \frac{1}{u^3} \psi_0(u) du = 2 \int_1^b \frac{1}{u^3} du \int_0^{+\infty} 1_{[0,u]}(y) y^2 \nu(dy) \\
&= 2 \int_0^{+\infty} y^2 \left(\int_{1 \vee y}^{b \vee y} \frac{du}{u^3} \right) \nu(dy) = \int_0^{+\infty} y^2 \left(\frac{1}{(1 \vee y)^2} - \frac{1}{(b \vee y)^2} \right) \nu(dy) \\
&= \left(1 - \frac{1}{b^2}\right) \int_0^1 y^2 \nu(dy) + \int_1^b y^2 \left(\frac{1}{y^2} - \frac{1}{b^2} \right) \nu(dy) \\
&= \int_1^b \nu(dy) + \int_0^1 y^2 \nu(dy) - \frac{1}{b^2} \int_0^b y^2 \nu(dy).
\end{aligned}$$

We have the following estimate, for any $b \in (1, \infty)$,

$$0 \leq f_0(b) \leq 2 \int_1^{+\infty} \log(y) \nu(dy) + 2 \log(b) \nu((b, +\infty)) + \int_1^{+\infty} \nu(dy) + \int_0^1 y^2 \nu(dy).$$

Setting $C = 2 \int_1^{+\infty} \log(y) \nu(dy) + \int_1^{+\infty} \nu(dy) + \int_0^1 y^2 \nu(dy) < +\infty$, we find

$$f_0(b) \leq C + 2 \frac{\log(b)}{\log(b)} \int_b^{+\infty} \log(y) \nu(dy) \leq 3C, \quad b \geq 1.$$

Note that (2.16) is equivalent to

$$\sum_{n \geq 1} \frac{1}{\gamma_n} f_0(e^{\gamma_n}) \leq 3C \sum_{n \geq 1} \frac{1}{\gamma_n} < +\infty.$$

The proof of the first part of the theorem is complete.

To show that there exists an invariant measure we first note that (according to [16, Theorem 17.5]) each one dimensional Ornstein-Uhlenbeck process (X_t^n) has an invariant measure μ_n which is the law of the random variable

$$\int_0^\infty e^{-\gamma_n u} \beta_n dZ_u^n$$

having characteristic function $\hat{\mu}_n(h) = \exp\left(-\int_0^\infty \psi(e^{-\gamma_n s} \beta_n h) ds\right)$, $h \in \mathbb{R}$.

Let us consider the product measure $\mu = \prod_{n \geq 1} \mu_n$ on $\mathbb{R}^{\mathbb{N}}$. This is the law of the $\mathbb{R}^{\mathbb{N}}$ -random variable $\xi = (\xi_n)$, where

$$\xi_n = \int_0^\infty e^{-\gamma_n u} \beta_n dZ_u^n, \quad n \geq 1.$$

According to Lemma 2.3, ξ takes values in H if and only if the Lévy measures ν_n of ξ_n verify

$$\sum_{n \geq 1} \int_{\mathbb{R}} (1 \wedge y^2) \nu_n(dy) < +\infty.$$

This condition is equivalent to

$$\sum_{n \geq 1} \frac{1}{\gamma_n} \int_1^{+\infty} \left(\frac{1}{u^3} \psi_0(u) + \frac{1}{u} \psi_1(u) \right) du = \sum_{n \geq 1} \frac{1}{\gamma_n} \left(\sup_{b \geq 1} f_0(b) \right) < +\infty$$

which holds. This shows that $\mu(H) = 1$ and so μ is a Borel probability measure on H .

We will prove that μ is the unique invariant measure of X by showing that, for any $x \in H$,

$$\lim_{t \rightarrow \infty} X_t^x = \xi \quad (2.17)$$

in probability (see [5]). It is enough to prove (2.17) when $x = 0$. Let $X_t^0 = Y_t$ and fix any $\epsilon > 0$. By using characteristic function, one checks easily that the law of Y_t is the same as the one of $\int_0^t e^{-\gamma_n u} \beta_n dZ_u^n$. We find

$$a_t = \mathbb{P}(|Y_t - \xi|^2 > \epsilon) = \mathbb{P}\left(\sum_{n \geq 1} \beta_n^2 \left(\int_t^\infty e^{-\gamma_n u} dZ_u^n\right)^2 > \epsilon\right)$$

Now, for any $t > 0$, we consider new independent Lévy processes $(Z_r^{t,n})_{r \geq 0}$, where $Z_r^{t,n} = Z_{r+t}^n - Z_t^n$, $r \geq 0$, $n \geq 1$. For any $t > 0$, $\int_t^\infty e^{-\gamma_n u} dZ_u^n$ has the same law as

$$\int_0^\infty e^{-\gamma_n(t+s)} dZ_s^{t,n} = e^{-\gamma_n t} \int_0^\infty e^{-\gamma_n s} dZ_s^{t,n}$$

which coincides with the law of $e^{-\gamma_n t} \frac{\xi_n}{\beta_n}$. By using independence, for any $t > 0$, the law of $\sum_{n \geq 1} \beta_n^2 \left(\int_t^\infty e^{-\gamma_n u} dZ_u^n\right)^2$ coincides with the one of

$$\sum_{n \geq 1} e^{-2\gamma_n t} \xi_n^2.$$

Assume that $\gamma_n \geq \gamma_0 > 0$, $n \geq 1$. We have, for any $t > 0$,

$$a_t = \mathbb{P}\left(\sum_{n \geq 1} e^{-2\gamma_n t} \xi_n^2 > \epsilon\right) \leq \mathbb{P}\left(e^{-2\gamma_0 t} \sum_{n \geq 1} \xi_n^2 > \epsilon\right) = \mathbb{P}(|\xi|^2 > e^{2\gamma_0 t} \epsilon).$$

By Letting $t \rightarrow \infty$, we find $\lim_{t \rightarrow \infty} a_t = 0$. This proves (2.17) with $x = 0$ and concludes the proof. \square

Example 2.12. Consider the following linear stochastic heat equation on $D = [0, \pi]^d$ with Dirichlet boundary conditions

$$\begin{cases} dX(t, \xi) = \Delta X(t, \xi) dt + dZ(t, \xi), & t > 0, \\ X(0, \xi) = x(\xi), & \xi \in D, \\ X(t, \xi) = 0, & t > 0, \quad \xi \in \partial D, \end{cases} \quad (2.18)$$

where Z is a cylindrical Lévy process with respect to the basis of eigenfunctions of the Laplacian Δ in $H = L^2(D)$ (with Dirichlet boundary conditions). The eigenfunctions are

$$e_j(\xi_1, \dots, \xi_d) = (\sqrt{2/\pi})^d \sin(n_1 \xi_1) \cdots \sin(n_d \xi_d), \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,$$

$j = (n_1, \dots, n_d) \in \mathbb{N}^d$. The corresponding eigenvalues are $-\gamma_j$, where $\gamma_j = (n_1^2 + \dots + n_d^2)$. The operator $A = \Delta$ with $D(A) = H^2(D) \cap H_0^1(D)$ verifies Hypothesis 2.1. \square

3 Irreducibility

We start with a simple lemma, which we prove for the reader convenience.

Lemma 3.1. *Let us consider a sequence (ξ_n) of independent real random variables, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

$$\sum_{n \geq 1} \xi_n^2 < +\infty, \quad \mathbb{P} - a.s..$$

If each ξ_n has full support in \mathbb{R} , then the random variable $\xi = (\xi_n)$ has full support in the Hilbert space l^2 .

Proof. We fix an arbitrary ball $B \subset l^2$, $B = B(y, r)$ with center in $y = (y_k) \in l^2$ and radius $r > 0$. Using independence, we find

$$\begin{aligned} \mathbb{P}\left(\sum_{k \geq 1} (\xi_k - y_k)^2 < r^2\right) &\geq \mathbb{P}\left(\sum_{k=1}^N (\xi_k - y_k)^2 < \epsilon, \sum_{k > N} (\xi_k - y_k)^2 < r^2 - \epsilon\right) \\ &\geq \mathbb{P}\left(\sum_{k=1}^N (\xi_k - y_k)^2 < \epsilon\right) \mathbb{P}\left(\sum_{k > N} (\xi_k - y_k)^2 < r^2 - \epsilon\right). \end{aligned}$$

Now we use that each ξ_n has full support in \mathbb{R} . This implies that, for any $N \in \mathbb{N}$, $\epsilon > 0$, $\mathbb{P}\left(\sum_{k=1}^N (\xi_k - y_k)^2 < \epsilon\right) > 0$. Since $\mathbb{P}\left(\sum_{k > N} (\xi_k - y_k)^2 < r^2 - \epsilon\right) \rightarrow 1$, as $N \rightarrow \infty$, the assertion follows. \square

We need the following result, which is a consequence of [16, Theorem 24.10].

Theorem 3.2. *Consider a symmetric infinitely divisible law μ on \mathbb{R} . If the support of its Lévy measure M contains 0 (i.e., for any $\delta > 0$, $M((-\delta, \delta)) > 0$), then the support of μ is \mathbb{R} .*

Proof. Arguing as in the proof of Proposition 2.2 we get that M is symmetric. Therefore, the support of M , which contains 0, has non-empty intersection with $(0, +\infty)$ and with $(-\infty, 0)$. By assertion (ii) in [16, Theorem 24.10], we get that the support of μ is \mathbb{R} . \square

Now we prove irreducibility of solutions to (1.1).

Theorem 3.3. *Assume Hypotheses 1.1 and 2.1. Moreover, suppose that the support of the Lévy measure ν given in (2.2) contains 0.*

Then, for any $x \in H$, the OU process (X_t^x) given in (2.6) is irreducible, that is, for any open ball $B \subset H$, $t > 0$, we have $\mathbb{P}(X_t^x \in B) > 0$.

Proof. According to Lemma 3.1, it is enough to prove that the one dimensional Ornstein-Uhlenbeck processes, starting from 0,

$$Y_t^n = Z_A^n(t) = \int_0^t e^{-\gamma_n(t-s)} \beta_n dZ_s^n, \quad n \in \mathbb{N}, \quad t > 0,$$

are irreducible. To this purpose, we fix $t > 0$ and denote by μ_n the symmetric and infinitely divisible law of Y_t^n , having Lévy measure $\tilde{\nu}_n$ of the form (2.11).

Using the fact that μ_n is symmetric and Theorem 3.2, to show that the support of μ_n is full in \mathbb{R} , we need to check that the support of $\tilde{\nu}_n$ contains 0. This follows easily from the assumption on ν and formula (2.11). \square

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