ON BOUNDED SOLUTIONS TO CONVOLUTION EQUATIONS

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Abstract: Periodicity of bounded solutions for convolution equations on a separable abelian metric group G is established and related Liouville type theorems are obtained. A non-constant Borel and bounded harmonic function is constructed for an arbitrary convolution semigroup on any infinite dimensional separable Hilbert space, generalizing a classical result by Goodman [14].

1 Introduction

Let μ be a probability measure defined on the σ-algebra $\mathcal{B}(G)$ of Borel subsets of a separable abelian metric group G, with the group operation "+". This group might be non-locally compact. The paper is concerned with convolution equations of the type

$$f * \mu(x) := \int_G f(x-y)\mu(dy) = f(x), \quad x \in G, \tag{1.1}$$

where $f : G \to \mathbb{R}$ is a Borel and bounded function, written shortly $f * \mu = f$.

Our aim is to investigate bounded Borel solutions f to (1.1). These functions are also called bounded μ-harmonic functions, see [6]. Special attention will be paid to the case when G is a real separable Hilbert space.

Convolution equations arise naturally in several areas of pure and applied mathematics like harmonic analysis, see [6], [4] and [15], the theory of Markovian semigroups, see [9], [1] and [11], and the renewal theory, see [10].

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Define the shift μ_h of the measure μ by the element $h \in G$, through the formula:

$$\mu_h(A) = \mu(A - h), \quad A \in \mathcal{B}(G).$$

Consider the set $M_\mu \subset G$ of all elements $h \in G$ such that measures μ_h are absolutely continuous with respect to μ. The set M_μ has been introduced in [12] and investigated, for instance, in [23], [22] and [2]. Our main result, see Theorem 2.5, shows that each Borel and bounded solution f to (1.1) is periodic with periods in M_μ i.e.,

$$f(x + h) = f(x), \quad x \in G, \ h \in M_\mu. \tag{1.2}$$

The theorem holds in the more general setting of measurable abelian groups, see Remark 2.11; moreover the set M_μ can be replaced by a larger set E_μ, see (2.1).

The result seems particularly useful in infinite dimensions, when G is a separable Hilbert space H and there is no a reference Haar measure $\mathcal{L}$. In Section 3.1 we consider an application to $\alpha-$stable measures μ on H, $\alpha \in (0, 2]$.

A related equation,

$$\nu * \mu = \nu, \tag{1.3}$$

on a locally compact group G, with unknown a non-negative measure ν, was the subject of a classical paper [3] by Choquet and Deny. Under suitable assumptions on ν, see Section 2.3, they proved that ν is periodic with periods in the subgroup generated by the support S_μ of μ (i.e., S_μ is the smallest closed set of G on which μ is concentrated). Within the class of locally compact groups, this result and related Liouville theorems have been extended in several directions, see [6], [15], [4], [20] and references therein.

Even when G is locally compact, our result does not follow from [3], see Remark 2.8. Moreover (1.2) does not hold if M_μ is replaced by S_μ. On the other hand, as a consequence of our main result, in Proposition 2.12 we obtain a version of the Choquet-Deny theorem concerning (1.3), which holds on metric groups (replacing the subgroup generated by S_μ with a smaller subgroup which contains M_μ). We also show in Theorem 2.9 that uniformly continuous and bounded solutions to (1.1) are periodic with periods in S_μ (see also Corollary 3.3.2 in [19]).

Equation (1.1) and related Liouville type theorems on a separable Hilbert space H are considered in Section 3. Let $(\mu_t)_{t \geq 0}$ be a convolution semigroup of probability measures on H. A function $f \in B_b(H)$ such that, for any $t \geq 0$,

$$f * \mu_t(x) = f(x), \quad x \in H, \tag{1.4}$$

is called a bounded harmonic function with respect to (μ_t). In Theorem 3.3 we prove that if the space H is infinite dimensional then one can always construct a discontinuous non-constant solution f to (1.4). This result generalises a well known example of Goodman [14], concerning the case when (μ_t) are Gaussian measures. The Goodman result shows clearly that (1.2) can not hold if M_μ is replaced by S_μ. This was one motivation for us to investigate (1.1).

Theorem 3.3 also implies that Markovian convolution semigroups associated to (μ_t) are never strong Feller in infinite dimensions, cf. [7].

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2 Convolution equations on metric groups

Let G be a separable abelian metric group with group operation indicated by $+$, see [16] and [22]. For A, $B \subset G$, we set

$$A + B = \{x + y,\ x \in A,\ y \in B\}, \quad -A = \{-x,\ x \in A\}.$$

Moreover $\mathrm{Gr}(A)$ denotes the smallest subgroup containing A and $\overline{A}$ the closure of A. The indicator function of a set $F \subset G$ will be indicated by 1_F.

When G is complete, we call it an abelian Polish group. When G is a real separable Hilbert space, we denote it by H. The inner product of H is then $\langle \cdot, \cdot \rangle$ and its norm $|\cdot|$.

The probability measures on G we consider will be always Borel probability measures on the σ-algebra $\mathcal{B}(G)$. Let σ be a probability measure on G. The *support* of σ is denoted by S_σ; it is the smallest closed set in G which has measure 1 with respect to σ.

With $\tilde{\sigma}$ we indicate its *reflection measure* with respect to 0, see [21, Chapter 1], i.e. $\tilde{\sigma}(A) = \sigma(-A)$, for any $A \in \mathcal{B}(G)$. The probability measure σ is called *symmetric* if $\sigma = \tilde{\sigma}$, i.e. $\sigma(A) = \sigma(-A)$, $A \in \mathcal{B}(G)$.

For μ and ν probability measures on G, the *convolution measure* $\mu * \nu$ is defined by $\mu * \nu(A) := \int_G \mu(A - x)\nu(dx)$, $A \in \mathcal{B}(G)$, see for instance [16] or [22]. Note that the operation $*$ is commutative and associative.

We write $\mu^n = \mu * \cdots * \mu$ ($n-$times), $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of all positive integers). We also set $\mu^0 = \delta_0$, where δ_0 is the Dirac measure concentrated in 0.

By $B_b(G)$ we denote the Banach space of all real, Borel and bounded functions $f : G \to \mathbb{R}$, endowed with the supremum norm $\|\cdot\|_\infty$. If $g \in B_b(G)$, we set

$$g * \mu(x) = \int_G g(x - y)\mu(dy), \quad x \in G.$$

2.1 Admissible shifts

Let T_a, $a \in G$, be the *translation operator*, i.e., $T_a(x) = x + a$, $x \in G$, and denote by $T_a \circ \mu$ the image of a probability measure μ under T_a, i.e., $(T_a \circ \mu)(A) = \mu(A - a)$, $A \in \mathcal{B}(G)$. We also set $(T_a \circ \mu) = \mu_a$.

According to [12, page 449], $a \in G$ is called *an admissible shift* for μ if $T_a \circ \mu$ is absolutely continuous with respect to μ. Let us denote by M_μ the set of all admissible shifts. Note that $0 \in M_\mu$. Moreover it is known that M_μ is always a semigroup of G, see [12, page 450]. Since $M_{\tilde{\mu}} = -M_\mu$, where $\tilde{\mu}$ is the reflection measure of μ, see [23], we have that M_μ is a subgroup of G if μ is symmetric.

If G is locally compact and μ is equivalent to the Haar measure of G, then it is easy to show that $M_\mu = G$.

We introduce the set

$$E_\mu = \bigcup_{n \geq 0} M_{\mu^n}, \qquad \mu^n = \mu * \cdots * \mu \ \ (n - times)\ \ n \in \mathbb{N}. \tag{2.1}$$

We will need the following elementary result.

Proposition 2.1. *Let μ and ν be probability measures on G. Then*

$$M_\mu + M_\nu \subset M_{\mu * \nu}. \tag{2.2}$$

Moreover E_μ is a semigroup of G.

Proof. Take any $A \in \mathcal{B}(G)$ such that $\mu * \nu(A) = 0$ and $a \in M_\mu$, $b \in M_\nu$. It is enough to show that $(\mu * \nu)_{a+b}(A) = 0$. We have

$$(\mu * \nu)_{a+b}(A) = \int_G \int_G 1_A(x+y) p(x) q(y) \mu(dx) \nu(dy),$$

where p and q are the densities of μ_a and ν_b respectively. For any $N \in \mathbb{N}$, set $p_N = N \wedge p$ and $q_N = N \wedge q$. Then

$$\int_G \int_G 1_A(x+y) p_N(x) q_N(y) \mu(dx) \nu(dy) \leq N^2 \int_G \int_G 1_A(x+y) \mu(dx) \nu(dy) = 0.$$

Taking $N \to \infty$, we get the first assertion. This implies that E_μ is an increasing union of semigroups. The second assertion follows easily. ■

The next example shows that the inclusion in (2.2) can be strict.

Example 2.2. *There exists a probability measure μ on $\mathbb{R}^n$ such that $M_\mu = 0$ and $M_{\mu^2} = \mathbb{R}^n = E_\mu$.*

Define $\mu = \sum_{k \geq 1} p_k \nu_k$, where ν_k, are uniform distributions on the spheres centered in 0, of radiuses $k \in \mathbb{N}$, and $\sum_{k \geq 1} p_k = 1$, $p_k > 0$, $k \in \mathbb{N}$. Since the measures $\nu_k * \nu_l$ are absolutely continuous with respect to Lebesgue measure, see [10], and the density of $\nu_k * \nu_k$ is positive on the ball $B(0, 2k)$, we have $\mu * \mu = \sum_{k,l=1}^{\infty} p_k p_l \, \nu_k * \nu_l$, and $\mu * \mu$ has a positive density on $\mathbb{R}^n$. The assertion follows.

We compare now the set E_μ with the subgroup generated by the support S_μ of μ.

Proposition 2.3. *Let μ be a probability measure on G. Then $E_\mu \subset Gr(S_\mu)$.*

Proof. Fix any $h \in M_\mu$. It is straightforward to check that $S_\mu + h \subset S_\mu$.

Now take $x \in S_\mu$, $x \neq 0$ (if $S_\mu = \{0\}$, then μ is the Dirac measure concentrated in 0 and $M_\mu = \{0\}$ as well). We know that $x + h \in S_\mu \subset \mathrm{Gr}(S_\mu)$. Since also $-x \in \mathrm{Gr}(S_\mu)$, it follows that $h \in \mathrm{Gr}(S_\mu)$. We have proved that $M_\mu \subset \mathrm{Gr}(S_\mu)$.

For any $n \in \mathbb{N}$, one has $M_{\mu^n} \subset \mathrm{Gr}(S_{\mu^n}) = \mathrm{Gr}(S_\mu)$. Hence the assertion holds. ■

In general the sets M_μ and S_μ are different.

Example 2.4. *There exists a probability measure μ on $\mathbb{R}^n$ such that M_μ and S_μ are disjoint sets.*

Take $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, and $u \in \mathbb{R}^n$ such that $|u| = 1$. Consider the line $L = \{x \in \mathbb{R}^n : x = \lambda u + x_0, \text{ for } \lambda \in \mathbb{R}\}$.

Let μ be a probability measure on $\mathbb{R}^n$, concentrated on L, having a positive density with respect to the one dimensional Lebesgue measure. We have that $M_\mu = \{\lambda u\}_{\lambda \in \mathbb{R}}$ and $S_\mu = L$.

2.2 Characterization theorems

Theorem 2.5. *Let G be a separable abelian metric group. Let μ be a probability measure on G and let $E_\mu \subset G$ be defined in (2.1). Let $f \in B_b(G)$ be a μ-harmonic function. Then one has:*

$$f(x+a) = f(x), \quad x \in G, \;\; a \in Gr(E_\mu). \tag{2.3}$$

If, in addition, f is continuous on G and $Gr(E_\mu)$ is dense in G, then f is constant.

The proof uses the following result, see [8, Theorem 9, page 292],

Theorem 2.6. *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space (with ν positive measure). Let K be a bounded subset of $L^1(\Omega, \nu)$. Assume that, for each decreasing sequence $(E_n) \subset \mathcal{F}$ with empty intersection, the limit $\lim_{n\to\infty} \int_{E_n} f(s)\nu(ds) = 0$ is uniform with respect to $f \in K$. Then, for any sequence $(f_n) \subset K$, there exists a subsequence (f_{n_k}) which converges weakly in $L^1(\Omega, \nu)$.*

Proof of Theorem 2.5. We first define, similarly to [3], suitable auxiliary functions and then obtain the required characterization arguing by contradiction. Both steps are accomplished differently from [3]. In particular, instead of the Ascoli-Arzela theorem, we use arguments based on *L^1-weak compactness.*

Let us introduce $\tilde{f}$, $\tilde{f}(x) = f(-x)$, $x \in G$. The equation $f * \mu = f$ is equivalent to

$$\tilde{f} * \tilde{\mu} = \tilde{f},$$

where $\tilde{\mu}$ is the reflection measure of μ. Fix $a \in M_\mu$ and introduce the function

$$g(x) = \tilde{f}(x) - \tilde{f}(x + a).$$

It is clear that $g \in B_b(G)$ and $g * \tilde{\mu} = g$ on G. Let

$$2c = \sup_{x\in G} g(x)$$

and $(x_n) \subset G$ such that $g(x_n) \to 2c$ as $n \to \infty$. Consider the functions $g_n : G \to \mathbb{R}$,

$$g_n(x) = g(x + x_n), \quad x \in G.$$

Each $g_n \in B_b(G)$ and solves the convolution equation (2.3). Now we set $L^1 = L^1(G, \mu)$ and use L^1-weak convergence ($L^\infty(G, \mu)$ is identified with the topological dual of L^1). The proof proceeds in some steps.

Step I. The sequence (g_n) is relatively L^1-weak sequentially compact.

We apply Theorem 2.6. To this purpose note that (g_n) is bounded in L^1 and moreover, for any decreasing sequence $(E_k) \subset \mathcal{B}(G)$, with empty intersection, one has:

$$\sup_{n\geq 0} \Big| \int_{E_k} g_n(y)\mu(dy) \Big| \leq 2\|f\|_\infty \mu(E_k),$$

which tends to 0 as $k \to \infty$. Hence, possibly passing to a subsequence, still denoted by (g_n), we know that there exists $g_0 \in L^1$ such that, for any $h \in L^\infty(G, \mu)$,

$$\int_G g_n(y)h(y)\,\mu(dy) \;\to\; \int_G g_0(y)h(y)\,\mu(dy), \quad \text{as } n \to \infty.$$

II step. The limit function $g_0 = 2c$, μ-a.s..

Note that, for $x \in G$,

$$g_n(x) = \int_G g_n(x - y)\tilde{\mu}(dy) = \int_G g_n(x + y)\mu(dy), \quad n \in \mathbb{N}. \tag{2.4}$$

Set $x = 0$ in (2.4). Using the L^1-weak convergence, we get

$$2c = \lim_{n\to\infty} g_n(0) = \int_G g_0(y)\mu(dy). \tag{2.5}$$

Now we prove that $g_0(x) \le 2c$, $\mu-$a.s. If this does not hold, then there exists $\epsilon > 0$ such that $B = \{x \in G : g_0(x) \ge 2c + \epsilon\}$ verifies $\mu(B) > 0$. But then, using that $g_n(x) \le 2c$, $x \in G$, we find

$$2c\mu(B) \ge \int_B g_n(y)\mu(dy) = \int_G g_n(y)\, I_B(y)\, \mu(dy), \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we infer a contradiction. By (2.5), we get the claim.

Step III. There exists a subsequence of (g_n)*, still denoted by* (g_n)*, which converges pointwise to* $2c$*,* $\mu-$*a.s.*

It is enough to show that (g_n) converges to $2c$ in probability (with respect to μ). To this purpose, we write, using that $g_n \le 2c$, for any $n \ge 1$,

$$\mu\Big(x \in G : |g_n(x) - 2c| > \epsilon\Big) \le \frac{1}{\epsilon}\int_G (2c - g_n(y))\mu(dy) \quad \to \; 0, \quad \text{as } n \to \infty.$$

Step IV. For any $x \in M_\mu$,

$$\lim_{n\to\infty} g_n(x) = 2c. \tag{2.6}$$

By (2.4) we have, for any $x \in M_\mu$,

$$g_n(x) = \int_G g_n(y)(T_x \circ \mu)(dy) = \int_G g_n(y)F^x(y)\mu(dy), \quad x \in M_\mu,\; n \in \mathbb{N}, \tag{2.7}$$

where T_x is the translation operator and F^x denotes the density of $(T_x \circ \mu)$ with respect to μ. Now we write, for any $M > 0$,

$$\begin{aligned} |g_n(x) - 2c| &= \Big|\int_G (g_n(y) - 2c)\,(T_x \circ \mu)(dy)\Big| \\ &\le \Big|\int_{\{y\,:\,|F^x(y)|>M\}} (g_n(y) - 2c)(T_x \circ \mu)(dy)\Big| + \Big|\int_{\{y\,:\,|F^x(y)|\le M\}} (g_n(y) - 2c)\, F^x(y)\mu(dy)\Big| \\ &\le 2(\|f\|_\infty + |c|)(T_x \circ \mu)\Big(\{|F^x(y)| > M\}\Big) + \Big|\int_G (g_n(y) - 2c)\, h^x(y)\, \mu(dy)\Big|, \end{aligned}$$

where $h^x(y) = F^x(y)\, I_{\{y\,:\,|F^x(y)|\le M\}}(y)$. For any $\epsilon > 0$, we can choose $M > 0$ and $n_0 \in \mathbb{N}$ large enough, such that $n \ge n_0$ implies $|g_n(x) - 2c| \le 2\epsilon$. The claim is proved.

Final Step. Recall that M_μ is a semigroup in G, see [12, page 450]. This fact and (2.6) imply that

$$g_0(ka) = 2c, \quad k \in \mathbb{N}.$$

Now we complete the proof similarly to Choquet-Deny [3]. For any integer m, there exists $\hat{n}$ such that

$$g_{\hat{n}}(ka) = \tilde{f}(x_{\hat{n}} + ka) - \tilde{f}(x_{\hat{n}} + (k-1)a) > c, \tag{2.8}$$

for $k = 1, \dots, m$. Summing (2.8) m-times, we get $\tilde{f}(x_{\hat{n}} + ma) - \tilde{f}(x_{\hat{n}}) > mc$.

Letting $m \to \infty$, we find that $c \le 0$, since $\tilde{f}$ is bounded. This means that $g(x) \le 0$, $x \in G$, i.e.,

$$\tilde{f}(x) \le \tilde{f}(x + a), \quad x \in G,$$

Repeating the previous argument with $-f$ instead of f, one has: $f(x) = f(x + a)$, $x \in G$. Thus (2.3) holds, for any $a \in M_\mu$. Now equation (1.1) implies that, for any $n \in \mathbb{N}$, $f * \mu^n(x) = f(x)$, $\; x \in G$. Hence (2.3) holds, for any $a \in E_\mu$. The assertion follows remarking that the set of all periods of a given real function on G is a subgroup of G. The proof is complete. ■

Remark 2.7. If μ is symmetric then $\mathrm{Gr}(E_\mu) = E_\mu$ and so the formulation of Theorem 2.5 simplifies. Indeed if μ is symmetric, then M_μ is a subgroup of G. By Proposition 2.1 we know that E_μ is an increasing union of subgroups. Hence E_μ is a group.

Remark 2.8. *Theorem 2.5 does not hold if we replace E_μ with the subgroup generated by the support S_μ of μ.*

Let $G = \mathbb{R}^d$ and take $\mathbb{Q}^d = \{q_n\}$ be the set of all points in $\mathbb{R}^d$ having rational coordinates. Let $(p_n) \subset \mathbb{R}_+$ be such that $\sum_{n\geq 1} p_n = 1$. Define $\mu = \sum_{n\geq 1} p_n \delta_{q_n}$, where δ_{q_n} are Dirac measures concentrated in q_n.

Take $f = 1_{\mathbb{Q}^d}$ be the indicator function of $\mathbb{Q}^d$. It is straightforward to check that $\mu * f(x) = f(x),\ x \in \mathbb{R}^n$. Moreover the support $S_\mu = \mathbb{R}^d$ and $M_\mu = \mathbb{Q}^d$.

Note that if $h \notin \mathbb{Q}^d$, then $f(x+h) = f(x)$ only if $x \notin A_h$, where $A_h = -h + \mathbb{Q}^d$ has Lebesgue measure 0.

If we restrict our attention to bounded μ-harmonic functions which are also uniformly continuous on G, then we can prove periodicity with respect to $\mathrm{Gr}(S_\mu)$. In the terminology of [4, page 396] the next result shows that any Polish abelian group has the Liouville property. We denote by $UC_b(G)$ the space of all uniformly continuous and bounded functions from G into $\mathbb{R}$.

Theorem 2.9. *Let G be a Polish abelian group. Let μ be a probability measure on G. Let $f \in UC_b(G)$ be a solution to $f * \mu = f$. Then,*

$$f(x+a) = f(x), \ \text{for any}\ x \in G,\ a \in Gr(S_\mu). \tag{2.9}$$

In Appendix we provide the complete proof. It uses the same arguments given in [3] but the next lemma is needed.

Lemma 2.10. *Let G be an abelian Polish group. There exists a subgroup $S_0 \subset G$, which is a countable union of compact sets and has the property that $\mu(S_0) = 1$.*

Proof. Chose first compact sets G_n such that $0 \in G_n$, $G_n \subset G_{n+1}$ and $\mu(G \setminus G_n) < 1/n$.

Define new compacts $F_n = (-G_n) \cup G_n$, $n \geq 1$ and finally set

$$K_1 = F_1,\ K_2 = F_2 + F_2,\ \ \ldots,\ \ K_n = F_n + \ldots + F_n\ (n - times),\ \ldots.$$

It is easy to check that $S_0 = \bigcup_{n\geq 1} K_n$, has all the required properties. ∎

Remark 2.11. Theorem 2.5 holds more generally, with the same proof, if G is a *measurable abelian group,* see [22, page 63]. A measurable space $(G, \mathcal{A})$ which is also a group (with additive notation) is said to be a measurable group if the group operations: $(x, y) \mapsto x + y$ and $x \mapsto -x$ are both measurable (on $G \times G$ one considers the product σ-algebra $\mathcal{A} \times \mathcal{A}$). In measurable groups the convolution of finite measures on $\mathcal{A}$ is naturally defined. Note that a separable metric group and a locally compact group, with $\mathcal{A}$ being the Borel σ-algebra, are both examples of measurable groups.

2.3 Connections with the Choquet-Deny theorem

It is interesting to compare our result with the remarkable theorem due to Choquet and Deny, mentioned in Introduction, valid in *locally compact groups* G, see [3]. Their theorem is concerned with the equation

$$\nu * \mu = \nu, \tag{2.10}$$

where μ is a given probability measure on G and unknown ν is a $\sigma-$finite Borel measure on G such that, for any *compact* set $K \subset G$, the Borel non-negative function:

$$G \to \mathbb{R}_+, \quad x \mapsto \nu(x-K) = 1_K * \nu(x) \text{ is finite and bounded on } G. \tag{2.11}$$

It turns out that ν is periodic with periods in the subgroup generated by the support S_μ of μ, i.e.

$$\nu(A+h) = \nu(A), \quad h \in Gr(S_\mu),\ A \in \mathcal{B}(G). \tag{2.12}$$

This result can be applied to the study of equation (1.1). Let us interpret function f as a density of a measure ν with respect to the Haar measure $\mathcal{L}$ of G: $f = \frac{d\nu}{d\mathcal{L}}$. Then (2.12) implies that, for any $h \in S_\mu$, $f(x+h) = f(x), \quad \mathcal{L} - a.s.$, where the set of $\mathcal{L}$-measure 0 depends, in general, on h, see Remark 2.8.

The following corollary of Theorem 2.5 can be regarded as a version of [3, Theorem 1] in the non-locally compact case.

Proposition 2.12. *Let μ be a given probability measure on a Polish abelian group G. Let ν be a $\sigma-$finite Borel measure on G satisfying the condition (2.11) for any compact set $K \subset G$. If $\nu * \mu = \nu$, then ν is periodic with periods in E_μ, see (2.12).*

Proof. Take first a compact set $K \subset G$ and consider the indicator function of K, i.e. 1_K. We have: $(1_K * \nu) * \mu = 1_K * \nu$. Indeed, by the Fubini theorem,

$$1_K * (\nu * \mu)(x) = \int_{G^2} 1_K(x-y-z)\nu(dy)\mu(dz) = (1_K * \nu) * \mu(x), \quad x \in G.$$

By Theorem 2.5 we get that $1_K * \nu(x+h) = 1_K * \nu(x)$, $x \in G$, $h \in E_\mu$. Hence, by taking $x = 0$, we get $\nu(h-K) = \nu(K)$, and so

$$\nu(h+K) = \nu(K), \text{ for any compact set } K \subset G,\ h \in E_\mu. \tag{2.13}$$

By the inner regularity of ν, for any Borel set $A \subset G$, with $\nu(A) < \infty$, there exists an increasing sequence of compact sets (K_n), such that $K_n \subset A$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \nu(K_n) = \nu(A)$. It follows that $\nu(h+A) = \nu(A)$, for any $h \in E_\mu$. The proof is complete. ∎

We do not know if the previous result holds when G is a Polish abelian group and E_μ is replaced by the subgroup generated by the support of μ.

Remark 2.13. Proposition 2.12 holds more generally, with the same proof, if G is a Hausdorff topological abelian group. In this case we assume that the measures μ and ν are both *Radon measures* (a non-negative Borel σ-finite measure γ on a Hausdorff topological space X is called Radon, if for each Borel set $B \subset X$, with $\gamma(B) < \infty$, for any $\epsilon > 0$, there exists a compact set $K \subset B$ such that $\gamma(B \setminus K) < \epsilon$, see for instance [22]).

3 Convolution equations on Hilbert spaces

3.1 The case of stable measures

Let $G = H$ be a real separable Hilbert space, Q a non-negative trace class operator on H and $\alpha \in (0, 2]$. A probability measure μ on H is said to be (α, Q)-stable, $\alpha \in (0, 2]$, centered at $x \in H$, if its characteristic function is

$$\hat{\mu}(h) := \int_H e^{i\langle h,y\rangle}\mu(dy) = \exp(i\langle x,h\rangle)\,\exp\Big(-\Big(\frac{\langle Qh,h\rangle}{2}\Big)^{\alpha/2}\Big), \quad h \in H, \tag{3.1}$$

see [16], [22], [21]. Such measures will be denoted by $N_\alpha(x,Q)$. Measures $N_2(x,Q)$ are Gaussian. In this case we also write $N(x,Q)$.

The following result extends Theorem 4.3.4 in [7].

Proposition 3.1. *Let $\mu = N_\alpha(x,Q)$, $\alpha \in (0,2]$. If $f \in B_b(H)$ solves equation (1.1) then*

$$f(y + Q^{1/2}a) = f(y), \quad y \in H, \;\; a \in H. \tag{3.2}$$

If, in addition, f is continuous and Q positive definite, then f is constant on H.

Proof. First we show the result for $\alpha = 2$.

Lemma 3.2. *Let $\mu = N(x,Q)$ and $\nu = N(y,S)$ be Gaussian measures on H. Then,*

$$M_{\mu*\nu} = M_\mu + M_\nu = Q^{1/2}H + S^{1/2}H. \tag{3.3}$$

In particular $E_\mu = M_\mu = Q^{1/2}H$.

Proof. It is well known that $M_\mu = Q^{1/2}H$, see [7]. Moreover $\mu * \nu = N(x+y, Q+S)$.

Define the linear operator $T : H \times H \to H$, $T(x,y) = Q^{1/2}x + S^{1/2}y$ (where as usual $\langle (x,y),(x',y')\rangle := \langle x,x'\rangle + \langle y,y'\rangle$). We check easily that

$$|(Q+S)^{1/2}h|^2 = |T^*h|^2, \;\; h \in H, \tag{3.4}$$

where T^* denotes the adjoint of T. By a classical duality argument, we have that $(Q+S)^{1/2}H = Q^{1/2}H + S^{1/2}H$. The proof is complete. ∎

Continuing the proof of the proposition note that by Lemma 3.2, $Q^{1/2}H = E_{N_2}$. Thus Theorem 2.5 gives the first claim. The second one follows from the density of $Q^{1/2}H$ in H when Q is non-degenerate.

Let us consider now $\alpha \in (0,2)$ and set $N_\alpha = N_\alpha(x,Q)$. We show that $Q^{1/2}H \subset M_{N_\alpha}$, $\alpha \in (0,2]$. For this we use subordination. Let ν_α by an $\alpha-$stable distributions ν^α on $[0,+\infty)$, with the Laplace transform given by

$$\int_0^\infty e^{-\lambda s}\nu^\alpha(ds) = e^{-\,(\lambda)^{\alpha/2}}, \;\; \lambda > 0.$$

It is easy to check that

$$N_\alpha(B) := \int_0^\infty N_2(x,sQ)(B)\,\nu^\alpha(ds), \quad B \in \mathcal{B}(H). \tag{3.5}$$

Take $A \in \mathcal{B}(H)$ such that $N_\alpha(A) = 0$, then, $N_2(x,sQ)(A) = 0$, for any $s \in S_{\nu^\alpha} = \mathbb{R}_+$. Let $g = Q^{1/2}h$, for some $h \in H$. By the absolute continuity of the Gaussian measures

$$(T_g \circ N_\alpha)(A) = N_\alpha(A+g) = \int_0^\infty N_2(x-g, sQ)(A)\nu^\alpha(ds) = 0.$$

Hence, $Q^{1/2}(H) \subset M_{N_\alpha}$. By Theorem 2.5 we get the claim. ∎

For informations about the set of all admissible shifts for general α-stable measures we refer to [2] and [23].

3.2 Liouville type theorems on Hilbert spaces

Let $\mu_t,\ t \geq 0$, be a convolution semigroup of measures on a real separable Hilbert space H. This means that $\mu_t * \mu_s = \mu_{t+s}$, $t,\ s \geq 0$, μ_0 is the Dirac measure concentrated in 0 and μ_t is weakly continuous at $t = 0$. Let P_t be the Markovian convolution semigroup determined by $\mu_t,\ t \geq 0$,

$$P_t f(x) = \int_H f(x-y)\mu_t(dy) = f * \mu_t(x), \quad x \in H,\ t \geq 0, f \in B_b(H), \tag{3.6}$$

see [16], [22], [13] and [21] for more information on convolution semigroups and Lévy processes. A function $h \in B_b(H)$ is said to be a *bounded harmonic function* for P_t, briefly a BHF for P_t, see [1], [9], [17] and [18] if

$$P_t h = h, \quad t \geq 0. \tag{3.7}$$

In particular, when P_t is a compound convolution semigroup, i.e., $P_t f(x) = e^{-\lambda t} \sum_{k\geq 0} \frac{(t\lambda)^k}{k!}\Big(f * (\nu)^k\Big)(x)$, where $\lambda > 0$ and ν is a given probability measure on H, one has that h is a BHF for P_t if and only if h is a bounded ν-harmonic function.

We present now a Liouville type theorem about BHFs for convolution semigroups. The first part is a consequence of Theorem 2.5. The second one is a generalization of a surprising result obtained by Goodman [14]. It states that in infinite dimensions there exist non-constant BHFs for the heat semigroup (see also [7], Section 4.3.1).

Theorem 3.3. *1) Let P_t be the Markovian semigroup (3.6) on a separable Hilbert space H and let*

$$\Gamma = Gr\Big(\bigcup_{t\geq 0} M_{\mu_t}\Big). \tag{3.8}$$

Then each BHF h for P_t is periodic with periods in Γ. If h is continuous and $\overline{\Gamma} = H$, then h is constant.

2) For arbitrary semigroups (3.6) there exists a non-constant BHF ϕ if H is infinite dimensional.

Proof. 1) It follows from Theorem 2.5. We only note that, by Proposition 2.1, one has: $Gr\Big(\bigcup_{t\geq 0} E_{\mu_t}\Big) = \Gamma$.

2) We use a probabilistic representation of convolution semigroups, see [13].

There exists a Lévy process (Z_t) on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, with values in H, such that the law of each Z_t is μ_t, $t \geq 0$. The process (Z_t) can be represented as

$$Z_t = at + \eta_t + \xi_t, \quad t \geq 0, \tag{3.9}$$

where $a \in \mathbb{R}^n$, (η_t) is a square integrable martingale and (ξ_t) is a compound Poisson process. Moreover the processes (η_t) and (ξ_t) are independent and so in particular

$$P_t f(x) = \mathbb{E} f(x - \xi_t - \eta_t - at) = f * \nu_t * r_t(x), \tag{3.10}$$

where ν_t is the law of $at + \eta_t$ and r_t the law of ξ_t. Thus it is enough to construct a non-constant function ϕ such that

$$\phi * \nu_t = \phi \ \text{ and } \ \phi * r_t = \phi, \quad t \geq 0. \tag{3.11}$$

Let us first consider $\eta_t + at$ with law ν_t. Remark that there exists a non-negative trace class operator $Q : H \to H$, such that this holds: $\langle Qh, k\rangle = \frac{1}{t}\mathbb{E}(\langle \eta_t, h\rangle\langle \eta_t, k\rangle)$, $t > 0$, $h, k \in H$.

Let us choose an orthonormal basis (e_k) in H, such that $Qe_k = \lambda_k e_k$. We have that $\sum_{k\geq 1}\lambda_k < \infty$. Let (α_k) be a sequence of positive numbers, diverging to $+\infty$, such that

$$\sum_{k\geq 1}\lambda_k\alpha_k + \sum_{k\geq 1} a_k^2\alpha_k < \infty, \quad a_k = \langle a, e_k\rangle, \;\; k \in \mathbb{N}, \tag{3.12}$$

and define the linear subspace K,

$$K = \{x \in H \,:\, g(x) < \infty\}, \quad g(x) = \sum_{k\geq 1} x_k^2\alpha_k, \;\; x \in H, \; x_k = \langle x, e_k\rangle. \tag{3.13}$$

Since (α_k) diverges, one has that K is *strictly* contained in H. Moreover $a \in K$ by construction. It turns out that the law of $\eta_t + at$ is concentrated on K, for any $t \geq 0$. Indeed one has:

$$\mathbb{E}g(\eta_t + at) = \sum_{k\geq 1}\alpha_k\langle \eta_t, e_k\rangle^2 + t\sum_{k\geq 1} a_k^2\alpha_k < \infty$$

and so $\eta_t + at$ is almost surely in K, for any $t \geq 0$. Note that

$$(I_K) * \nu_t(x) = \int_H I_K(x-y)\nu_t(dy) = \nu_t(x - K) = I_K(x), \;\; x \in H, t \geq 0.$$

Indeed if $x \notin K$, then $x - k \notin K$, for any $k \in K$. This gives that I_K is a non-constant BHF for the convolution semigroup determined by the process $(\eta_t + at)$.

Let us consider the remainder compound Poisson process (ξ_t), see (3.9). Denote by $\lambda > 0$ its intensity, by ν its Lévy measure and by S_t the associated convolution semigroup. One has:

$$S_t f(x) = \mathbb{E}f(x - \xi_t) = e^{-\lambda t}\sum_{k\geq 0}\frac{(t\lambda)^k}{k!}\Big(f * (\nu)^k\Big)(x), \tag{3.14}$$

where $\mathbb{E}(e^{i\langle \xi_t, h\rangle}) = e^{-t\psi(h)}$, $h \in H$, $t \geq 0$, and $\psi(h) = \int_H (e^{i\langle x,h\rangle} - 1)\nu(dx)$. As already noted, $h \in B_b(H)$ is a BHF for S_t if and only if $h * \nu = h$.

Let us first construct a non-constant bounded ν-harmonic function h. Let us introduce $\lambda'_k = \mathbb{E}(e^{-|U|}\langle U, e_k\rangle^2)$, where $U : \Omega \to H$ is a random variable with law ν. It is clear that

$$\sum_{k\geq 1}\lambda'_k = \mathbb{E}(e^{-|U|}|U|^2) < \infty.$$

Take a diverging sequence of positive real numbers (α'_k), such that $\sum_{k\geq 1}\lambda'_k\alpha'_k < \infty$ and define the linear subspace K', $K' = \{x \in H \,:\, \sum_{k\geq 1} x_k^2\alpha'_k < \infty\}$. One has that h is a non-costant BHF for S_t.

To finish the proof, define $\tilde{\alpha}_k = \min(\alpha_k, \alpha'_k)$, and introduce the new subspace $\tilde{K} = \{x \in H\,, \sum_{k\geq 1} x_k^2\tilde{\alpha}_k < \infty\}$. Repeating the previous arguments, we find that $\phi = I_{\tilde{K}}$ is non-constant and verifies (3.11). This completes the proof. ∎

We recall that a Markovian semigroup P_t, acting on $B_b(H)$, is called *strong Feller* if $P_t f$ is continuous on H, for any $f \in B_b(H)$, $t > 0$, see for instance [7].

Corollary 3.4. *Markovian convolution semigroups P_t, given by (3.6) on an infinite dimensional Hilbert space H, are never strong Feller.*

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Appendix: Proof of Theorem 2.9

Fix $a \in G$ and consider the function $g : G \to \mathbb{R}$,

$$g(x) = f(x) - f(x-a), \quad x \in G.$$

Now $g \in UC_b(G)$ and $g * \mu = g$ on G. Let

$$2c = \sup_{x \in G} g(x)$$

and $(x_n) \subset G$ such that $g(x_n) \to 2c$ as $n \to \infty$. We introduce also the functions $g_n : G \to \mathbb{R}$,

$$g_n(x) = g(x + x_n), \quad x \in G.$$

Again each g_n solves (1.1). Moreover, since $g \in UC_b(G)$, (g_n) is a relatively compact sequence in $C(K)$ for each compact set $K \subset G$.

Now we consider a *subgroup* $S_0 \subset G$, countable union of compact sets K_m, $m \in \mathbb{N}$, and having the property that

$$\mu(S_0) = 1,$$

see Lemma 2.10. Since (g_n) is relatively compact on each $C(K_m)$, for any $m \geq 1$, we get, by a diagonal procedure, that there exists a subsequence of (g_n), still denoted by (g_n), which converges to a continuous and bounded function $g_0 : S_0 \to \mathbb{R}$, uniformly on each K_m. In particular,

$$\lim_{n \to \infty} g_n(x) = g_0(x), \quad x \in S_0.$$

Passing to the limit in

$$g_n(x) = \int_{S_0} g_n(x-y)\mu(dy), \quad x \in S_0,$$

as $n \to \infty$, by the dominated convergence theorem, we infer:

$$g_0(x) = \int_{S_0} g_0(x-y)\mu(dy), \quad x \in S_0. \tag{3.15}$$

Setting $x = 0$, we get

$$2c = g_0(0) = \int_{S_0} g_0(-y)\mu(dy) = \int_{S_0} g_0(y)\tilde{\mu}(dy),$$

where $\tilde{\mu}$ is the reflection of μ with respect to 0. From the previous identity, using that g_0 is continuous on S_0 and that $g_0 \leq 2c$, we obtain

$$g_0(x) = 2c, \quad x \in S_1 = (S_0 \cap S_{\tilde{\mu}}),$$

where $S_{\tilde{\mu}} = -S_\mu$ denotes the support of $\tilde{\mu}$. Note that $\tilde{\mu}(S_1) = 1$.

Thanks to (3.15), we get

$$2c = g_0(x) = \int_{S_1} g_0(x+y)\tilde{\mu}(dy), \quad x \in S_1.$$

It follows that $g_0(x+y) = 2c$, x, $y \in S_1$. Using an induction argument, we find that

$$g_0(z) = 2c, \quad z \in S_2 = S_1 \cup (S_1 + S_1) \cup \ldots \cup (S_1 + \ldots + S_1) \cup \ldots, \tag{3.16}$$

where S_2 is the *semigroup* generated by S_1. Note that $S_2 \subset S_0$ (in fact S_0 is a group).

Now we choose the initial point a in S_1. By (3.16), we deduce that

$$g_0(ka) = 2c, \quad k \geq 1.$$

Hence for any integer m, there exists $\hat{n}$ such that

$$g_{\hat{n}}(ka) = f(x_{\hat{n}} + ka) - f(x_{\hat{n}} + (k-1)a) > c, \tag{3.17}$$

for $k = 1, \ldots, m$. Summing (3.17) m-times, we get

$$f(x_{\hat{n}} + ma) - f(x_{\hat{n}}) > mc.$$

Letting $m \to \infty$, we find that $c \leq 0$, since f is bounded. This means that $g(x) \leq 0$, $x \in G$, i.e.

$$f(x) \leq f(x-a), \quad x \in G, \ a \in (S_0 \cap S_{\tilde{\mu}}).$$

Repeating the previous argument with $-f$ instead of f, one has: $f(x) = f(x-a)$, $x \in G$, $a \in (S_0 \cap S_{\tilde{\mu}})$.

Since $\tilde{\mu}(S_0 \cap S_{\tilde{\mu}}) = \mu(S_0 \cap S_\mu) = 1$, we have that $(S_0 \cap S_{\tilde{\mu}})$ is dense in $S_{\tilde{\mu}}$ (indeed by definition $S_{\tilde{\mu}}$ is the smallest closed set on which $\tilde{\mu}$ is concentrated). Using the continuity of f one gets

$$f(x) = f(x-a), \quad x \in G, \ a \in S_{\tilde{\mu}}.$$

To finish the proof, one remarks that the set of all periods of a given real function on G is a subgroup of G. ∎