

Global Schauder estimates for a class of degenerate Kolmogorov equations

ENRICO PRIOLA ¹

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Abstract: We consider a class of possibly degenerate second order elliptic operators \mathcal{A} on \mathbb{R}^n . This class includes hypoelliptic Ornstein-Uhlenbeck type operators having an additional first order term with unbounded coefficients. We establish global Schauder estimates in Hölder spaces both for elliptic equations and for parabolic Cauchy problems involving \mathcal{A} . The considered Hölder spaces are defined with respect to a possibly non-euclidean metric related to the operator \mathcal{A} . We also prove sharp L^∞ -estimates on the spatial derivatives of the associated diffusion semigroup involving Hölder norms.

1 Introduction

Let us consider the following possibly degenerate second order elliptic operator \mathcal{A} on \mathbb{R}^n :

$$\begin{aligned} \mathcal{A}u(x) &= \frac{1}{2}\text{Tr}(QD^2u(x)) + \langle Ax, Du(x) \rangle + \langle F(x), Du(x) \rangle \\ &= \mathcal{A}_0u(x) + \langle F(x), Du(x) \rangle, \quad x \in \mathbb{R}^n. \end{aligned} \tag{1.1}$$

Here Q and A are $n \times n$ real matrices, Q is symmetric and non-negative definite, $\text{Tr}(\cdot)$ denotes the trace and $\langle \cdot, \cdot \rangle$ the inner product in \mathbb{R}^n . Moreover $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a regular possibly unbounded vector field. Degenerate Kolmogorov operators like \mathcal{A} arise in Kinetic Theory and in Mathematical Finance (see, for instance, [6, 7] and the references therein). Moreover, the operator \mathcal{A} contains in the special case of $F = 0$ the intensively studied possibly degenerate Ornstein-Uhlenbeck operator \mathcal{A}_0 .

The aim of the present paper is to prove global Schauder estimates for elliptic equations and parabolic Cauchy problems involving the operator \mathcal{A} . We obtain optimal regularity results in Hölder spaces for both

$$\lambda u(x) - \mathcal{A}u(x) = f(x), \quad x \in \mathbb{R}^n, \quad \text{and} \tag{1.2}$$

$$\begin{cases} \partial_t v(t, x) = \mathcal{A}v(t, x) + H(t, x), & t \in (0, T], x \in \mathbb{R}^n, \\ v(0, x) = g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

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where $\lambda > 0$ and the functions f, g and H are assigned. These results are deduced by sharp L^∞ -estimates on the spatial derivatives of the solution of (1.3) when $H = 0$, involving Hölder norms of the initial datum g .

Let us collect our assumptions on the operator \mathcal{A} (compare with [25]).

Hypothesis 1.1. (i) The symmetric matrix $Q = (q_{ij})_{i,j=1,\dots,n}$ is given by

$$Q = \begin{pmatrix} Q_0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ where } Q_0 \text{ is a positive definite } \tilde{p} \times \tilde{p}\text{-matrix, } 1 \leq \tilde{p} \leq n; \quad (1.4)$$

ν_1 and ν_2 stand for the smallest and the largest eigenvalue of Q_0 respectively ($0 < \nu_1 \leq \nu_2$);

(ii) the vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has the form $F(x) = (F_1(x), \dots, F_{\tilde{p}}(x), 0, \dots, 0)$, $x \in \mathbb{R}^n$, i.e., $F(x) \in \text{Im}(Q)$, for any $x \in \mathbb{R}^n$;

(iii) the non-zero coefficients of F , i.e., $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, \tilde{p}$, are Lipschitz continuous having continuous and bounded partial derivatives up to the third order on \mathbb{R}^n ;

(iv) there exists a non-negative integer k , such that the vectors

$$\{e_1, \dots, e_{\tilde{p}}, Ae_1, \dots, Ae_{\tilde{p}}, \dots, A^k e_1, \dots, A^k e_{\tilde{p}}\} \text{ generate } \mathbb{R}^n, \quad (1.5)$$

where $e_1, \dots, e_{\tilde{p}}$ are the first \tilde{p} elements of the canonical basis in \mathbb{R}^n ; we denote by k the *smallest* non-negative integer such that (1.5) holds (one has $0 \leq k \leq n - 1$).

Condition (1.5) can be also written as $\text{Rank}[Q^{1/2}, AQ^{1/2}, \dots, A^k Q^{1/2}] = n$. By the Hörmander condition on commutators, (1.5) is equivalent to the hypoellipticity of the operator $\mathcal{A}_0 - \partial_t$ in $(n + 1)$ variables (t, x_1, \dots, x_n) ; see [13]. Our operator \mathcal{A} has the form

$$\mathcal{A}u(x) = \frac{1}{2} \sum_{i,j=1}^{\tilde{p}} q_{ij} \partial_{x_i x_j}^2 u(x) + \sum_{i=1}^{\tilde{p}} F_i(x) \partial_{x_i} u(x) + \sum_{i,j=1}^n a_{ij} x_j \partial_{x_i} u(x), \quad x \in \mathbb{R}^n,$$

where the a_{ij} are the components of the matrix A and ∂_{x_i} and $\partial_{x_i x_j}^2$ are partial derivatives. Clearly, the operator \mathcal{A} is non-degenerate only when $\tilde{p} = n$ (this implies $k = 0$).

Let us describe our results for (1.2) and (1.3). In the elliptic equation (1.2) we assume that $f \in \mathcal{C}_d^\theta(\mathbb{R}^n)$, $\theta \in (0, 1)$, i.e., f is a real bounded function on \mathbb{R}^n , which is Hölder continuous with respect to a suitable possibly non-euclidean metric d related to \mathcal{A} . We show that (1.2) has a unique bounded distributional solution $u \in \mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$ and, moreover, that there exists a positive constant C , independent of f and u , such that $\|u\|_{2+\theta, d} \leq C \|f\|_{\theta, d}$. Note that this implies

$$\|u\|_0 + \sum_{i,j=1}^{\tilde{p}} \|\partial_{x_i x_j}^2 u\|_{\theta, d} \leq C \|f\|_{\theta, d},$$

where $\|u\|_0$ denotes the sup-norm of u (see Theorem 4.2). Concerning the Cauchy problem (1.3) we prove analogous parabolic Schauder estimates, assuming that $g \in \mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$ and $H(t, \cdot) \in \mathcal{C}_d^\theta(\mathbb{R}^n)$, uniformly in $t \in [0, T]$ (see Theorem 4.3). The metric d is mentioned in [29, page 11] and is related to certain distances associated to degenerate operators such as Hörmander's sum of squares of vector fields (see in particular the metric ρ_3 in [22, page 112]). Moreover, d is a special case of the parabolic pseudo-metric considered in [7] (see also [13]). We refer to Section 2 for a precise definition of the metric d . Here we give an example of d . Consider the following two-dimensional operator \mathcal{A} ,

$$\mathcal{A}u(x, y) = \frac{1}{2} \partial_{xx}^2 u(x, y) + F_1(x, y) \partial_x u(x, y) + (x + y) \partial_y u(x, y), \quad (x, y) \in \mathbb{R}^2, \quad (1.6)$$

which verifies Hypothesis 1.1 with $\tilde{p} = 1$ and $k = 1$. In this case, the metric d is given by $d(z, z') = |x - x'| + |y - y'|^{1/3}$, for any $z = (x, y)$ and $z' = (x', y') \in \mathbb{R}^2$.

Let's now examine related papers on Schauder estimates. A general theory of *local* regularity in Sobolev and Hölder spaces is available for degenerate operators which are sums of squares of vector fields (see in particular [8, 26, 11]). Local C^θ -estimates for operators more general than \mathcal{A} , in which also q_{ij} are variable and time-dependent, can be found in [7] (see also [13, 21, 23] and the references therein). Concerning *global* regularity results for solutions of possibly degenerate equations like (1.2) and (1.3) in spaces of continuous functions, we mention [16, 14, 15, 23, 27]. In [16] Schauder estimates are established for the Ornstein-Uhlenbeck operator \mathcal{A}_0 only assuming (1.4) and (1.5). In [14, 15] Schauder estimates are proved for Ornstein-Uhlenbeck type operators \mathcal{A}_0 when $F_i = 0$ but q_{ij} are not constant and can be unbounded; in [14, 15] it is assumed $k \leq 1$ in hypothesis (1.5). Uniform estimates for solutions to the Cauchy problem (1.3) involving \mathcal{A} with $H = 0$ are given in [25]; these are obtained without any restriction on k and are preliminary to the Schauder estimates of the present paper. In [27] Schauder estimates for \mathcal{A} are proved assuming $k \leq 1$ in (1.5) and imposing an additional hypothesis (which is not satisfied in (1.6)).

To prove elliptic Schauder estimates, one considers the function

$$u(x) = \int_0^{+\infty} e^{-\lambda t} P_t f(x) dt, \quad x \in \mathbb{R}^n, \quad (1.7)$$

where P_t is the diffusion Markov semigroup associated to \mathcal{A} (i.e., $v(t, x) = (P_t f)(x) = P_t f(x)$ provides the classical solution to (1.3) when $H = 0$ and $g = f$; see [25]). The function u is the unique bounded distributional solution to (1.2) (see Theorem 4.1). One proves global regularity properties for u by means of sharp L^∞ -estimates on the spatial partial derivatives of $P_t f$ involving the Hölder norm of f like

$$\|\partial_{x_i x_j x_r}^3 P_t f\|_0 \leq c \left(\frac{1}{t^{\frac{3-\theta}{2}}} + 1 \right) \|f\|_{\theta, d}, \quad t > 0, \quad i, j, r = 1, \dots, \tilde{p}$$

(see Theorem 3.3). The behaviour in t of such estimates as t tends to 0^+ is crucial. This is the basic idea indicated in [5] in order to study Schauder estimates for non-degenerate Kolmogorov operators. This method has been much used in recent papers also in combination with [17] (see [4, Chapter 1], [2, Chapter 6] and the references therein). In [16] the L^∞ -estimates have been proved using the explicit formula of the Ornstein-Uhlenbeck semigroup P_t associated to \mathcal{A}_0 . In [14, 15, 27] the uniform estimates are obtained by a priori estimates of Bernstein type combined with an interpolation result proved in [15, Lemma 5.1] when $k \leq 1$. Here we get the L^∞ -estimates involving Hölder norms by working directly on some probabilistic formulae for the spatial derivatives of $P_t f$ (which replace the explicit formulae used in [16]). Such formulae have been obtained in [25] using Malliavin Calculus (see also [3, 12, 9]).

We believe that the probabilistic approach used here could be useful to get sharp L^∞ -estimates involving Hölder norms also in other situations. Specifically, one could consider a degenerate Kolmogorov operator like \mathcal{A} in which the drift term $Ax + F$ is replaced by a C^∞ -vector field $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ having additional properties. G should have all bounded derivatives up to some order $k \in \mathbb{N}$ and, moreover, $e_1, \dots, e_{\tilde{p}}$ and G together with their commutators of length at most k should span \mathbb{R}^n at each point $x \in \mathbb{R}^n$. In this case global Schauder estimates are a largely open problem.

From the previous L^∞ -estimates we derive in a direct and relatively short way Schauder estimates for u (see Theorem 4.2). This does not require the interpolation method of [17] which has been used in [16, 4, 15, 2, 27]. Our direct approach can be applied to prove parabolic Schauder estimates as well (see Theorem 4.3). In order to study the parabolic Cauchy problem (1.3) one proceeds initially as in the elliptic case, replacing (1.7) with the variation of constant formula (see (4.5)). However, the parabolic Schauder estimates are more involved than the corresponding elliptic ones (see Remark 3.4). In particular, they require not only the L^∞ -estimates involving Hölder norms but also the hard estimate

$$\|P_t g\|_{2+\theta, d} \leq C \|g\|_{2+\theta, d},$$

for any $g \in C_d^{2+\theta}(\mathbb{R}^n)$, $t \geq 0$, where C is independent of t and g .

After some preliminaries contained in Section 2, in Section 3 we prove L^∞ -estimates for the spatial derivatives of $P_t f$ involving the Hölder norm of f . In Section 4 we show that (1.2) has a unique distributional solution and prove elliptic Schauder estimates using the results of Section 3. We also establish existence and uniqueness of space-distributional solutions to the parabolic Cauchy problem (1.3) and prove the parabolic Schauder estimates. In the final part of the paper we consider more general operators \tilde{A} with variable coefficients $q_{ij}(x)$. We require that the matrix $Q(x)$ has the form (1.4) where the $\tilde{p} \times \tilde{p}$ matrix $Q_0(x)$ is uniformly positive; moreover, we assume that q_{ij} are θ -Hölder continuous and that there exists $\lim_{x \rightarrow \infty} Q_0(x) = Q_0^\infty$. We obtain elliptic and parabolic Schauder estimates for \tilde{A} , using a well-known method based on maximum principle, a priori estimates and continuity method (compare with [16]). Further extensions of our results are given in Remark 5.4.

We will use the letter c or C with subscripts for finite positive constants whose precise value is unimportant; the constants may change from proposition to proposition.

2 Preliminaries and notation

We denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the euclidean norm and the standard inner product in \mathbb{R}^n and by $\|\cdot\|_L$ the operator norm in the Banach space $L(\mathbb{R}^n)$ of real $n \times n$ matrices. If X and Y are real Banach spaces, $L(X, Y)$ denotes the Banach space of all bounded and linear operators from X into Y endowed with the operator norm.

Let $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a mapping. We denote by $DG(x)$, $D^2G(x)$ and $D^3G(x)$ respectively the first, second and third Fréchet derivative of G at $x \in \mathbb{R}^n$ when they exist (if G also depends on t , we write $D_x G(t, x)$, $D_{xx}^2 G(t, x)$ and $D_{xxx}^3 G(t, x)$). We have $DG(x)[u]$, $D^2G(x)[u][v]$ and $D^3G(x)[u][v][w] \in \mathbb{R}^m$, for $u, v, w \in \mathbb{R}^n$. If G is bounded, we set $\|G\|_0 = \sup_{x \in \mathbb{R}^n} |G(x)|_{\mathbb{R}^m}$.

Recall that hypothesis (1.5) is known as the Kalman condition in control theory (see [31]). It is also equivalent to requiring that the following symmetric matrix Q_t ,

$$Q_t = \int_0^t e^{sA^*} Q e^{sA} ds \tag{2.1}$$

is positive definite for any $t > 0$ (here e^{sA} denotes the exponential matrix of A and A^* the adjoint matrix of A).

As in [16] we define an *orthogonal decomposition* of \mathbb{R}^n related to the Kalman condition (1.5). We consider the first \tilde{p} elements $\{e_1, \dots, e_{\tilde{p}}\}$ of the canonical basis in \mathbb{R}^n , $1 \leq \tilde{p} \leq n$, and introduce the subspace $V_0 = \text{Span}\{e_1, \dots, e_{\tilde{p}}\}$. Then set $V_m = \text{Im}(Q^{1/2}) +$

$\dots + \text{Im}(A^m Q^{1/2}) = \text{Span}\{e_1, \dots, e_{\tilde{p}}, Ae_1, \dots, Ae_{\tilde{p}}, \dots, A^m e_1, \dots, A^m e_{\tilde{p}}\}$, for $1 \leq m \leq k$. One has $V_m \subset V_{m+1}$ and $V_k = \mathbb{R}^n$. Let $W_0 = V_0$, W_1 be the orthogonal complement of V_0 in V_1 , W_m be the orthogonal complement of V_{m-1} in V_m , for $1 \leq m \leq k$. Defining the orthogonal projections E_m from \mathbb{R}^n onto W_m , one has $E_m(\mathbb{R}^n) = W_m$ and

$$\mathbb{R}^n = \bigoplus_{m=0}^k E_m(\mathbb{R}^n), \quad (2.2)$$

We complete $\{e_1, \dots, e_{\tilde{p}}\}$ in order to get an *orthonormal reference basis* $\{e_i\}_{i=1, \dots, n}$ in \mathbb{R}^n related to (2.2). This consists of generators of the subspaces and will be used throughout the paper (one can assume that $\{e_i\}$ is the canonical basis if $k \leq 1$, compare with [15, 27]). In the sequel D_i , D_{ij}^2 , and D_{ijr}^3 will denote respectively first, second and third partial derivatives with respect to $\{e_i\}$. Note that, writing the operator \mathcal{A} in the coordinates associated to the new basis, the second order term $\text{Tr}(QD^2)$ does not change. Define I_m as the set of indices i such that e_i spans $E_m(\mathbb{R}^n)$, $0 \leq m \leq k$. We have

$$I_0 = \{1, \dots, \tilde{p}\}.$$

The metric d associated to the operator \mathcal{A} is defined using the decomposition (2.2). One first introduces the *quasi-norm* $\|\cdot\|$, $\|x\| := \sum_{h=0}^k |E_h x|^{1/(2h+1)}$, $x \in \mathbb{R}^n$. Then we set

$$d(x, y) := \|x - y\| = \sum_{h=0}^k |E_h(x - y)|^{\frac{1}{2h+1}}, \quad x, y \in \mathbb{R}^n. \quad (2.3)$$

Let us introduce function spaces used to establish Schauder estimates. First we consider euclidean function spaces and then function spaces related to the metric d .

We denote by $B_b(\mathbb{R}^n)$ the Banach space of all Borel and bounded functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, endowed with the supremum norm $\|\cdot\|_0$; $C_b(\mathbb{R}^n)$ is the closed subspace of $B_b(\mathbb{R}^n)$ consisting of all *uniformly continuous and bounded* functions.

$C_b^j(\mathbb{R}^n)$, $j \in \mathbb{Z}_+$, $j \geq 1$, is the Banach space of all j -times differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, whose partial derivatives, $D_\alpha f$, $\alpha \in \mathbb{Z}_+^n$, are uniformly continuous and bounded on \mathbb{R}^n up to order j . This is a Banach space endowed with the norm $\|\cdot\|_j$, $\|f\|_j = \|f\|_0 + \sum_{|\alpha| \leq j} \|D_\alpha f\|_0$, $f \in C_b^j(\mathbb{R}^n)$. We set $C_b^\infty(\mathbb{R}^n) = \bigcap_{j \geq 1} C_b^j(\mathbb{R}^n)$. Moreover $C_0^\infty(\mathbb{R}^n)$ is the space of all functions $f \in C_b^\infty(\mathbb{R}^n)$ having compact support.

Fix $\theta \in (0, 1)$. The space $C_b^\theta(\mathbb{R}^n)$ stands for the Banach space of all θ -Hölder continuous and bounded functions on \mathbb{R}^n endowed with the norm $\|\cdot\|_\theta$, i.e., $\|f\|_\theta = \|f\|_0 + [f]_\theta$, $f \in C_b^\theta(\mathbb{R}^n)$, where $[f]_\theta = \sup_{z, w \in \mathbb{R}^n, z \neq w} \frac{|f(z) - f(w)|}{|z - w|^\theta} < \infty$. Moreover $C_b^{2+\theta}(\mathbb{R}^n) = \{f \in C_b^2(\mathbb{R}^n) : D_{ij}^2 f \in C_b^\theta(\mathbb{R}^n), i, j = 1, \dots, n\}$; it is a Banach space endowed with the norm $\|\cdot\|_{2+\theta}$, $\|f\|_{2+\theta} = \|f\|_2 + \sum_{i, j=1}^n \|D_{ij}^2 f\|_\theta$, $f \in C_b^{2+\theta}(\mathbb{R}^n)$. In a similar way, one defines the Banach space $C_b^{1+\theta}(\mathbb{R}^n)$. Next, we define function spaces related to the metric d .

Let $\gamma \in (0, 3)$ and γ *non-integer*. We define $\mathcal{C}_d^\gamma(\mathbb{R}^n)$ as the space of all functions $f \in C_b(\mathbb{R}^n)$ such that, for any $z \in \mathbb{R}^n$ and for any integer m , $0 \leq m \leq k$, the map:

$$x \mapsto f(z + x) \text{ belongs to } C_b^{\gamma/(2m+1)}(E_m(\mathbb{R}^n)),$$

with the $\|f(z + \cdot)\|_{\gamma/(2m+1)}$ bounded by a constant *independent* of z (identifying each subspace $E_m(\mathbb{R}^n)$ with $\mathbb{R}^{n(m)}$, where $n(m) = \dim[E_m(\mathbb{R}^n)]$, the euclidean function spaces $C_b^{\gamma/(2m+1)}(E_m(\mathbb{R}^n))$ are well-defined); $\mathcal{C}_d^\gamma(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_{\gamma, d}$,

$$\|f\|_{\gamma, d} := \sum_{m=0}^k \sup_{z \in \mathbb{R}^n} \|f(z + \cdot)\|_{C_b^{\gamma/(2m+1)}(E_m(\mathbb{R}^n))}, \quad f \in \mathcal{C}_d^\gamma(\mathbb{R}^n).$$

It is easy to see that if $\gamma \in (0, 1)$ and $f \in C_b(\mathbb{R}^n)$, then $f \in \mathcal{C}_d^\gamma(\mathbb{R}^n)$ if and only if f is γ -Hölder continuous with respect to the metric d , i.e.

$$[f]_{\gamma,d} = \sup_{x,y \in \mathbb{R}^n, x \neq y} |f(x) - f(y)| \|x - y\|^{-\gamma} < +\infty. \quad (2.4)$$

Moreover, an equivalent norm in $\mathcal{C}_d^\gamma(\mathbb{R}^n)$, $\gamma \in (0, 1)$, is $\|\cdot\|_0 + [\cdot]_{\gamma,d}$. One can also define spaces $\mathcal{C}_d^\alpha(\mathbb{R}^n)$ for general $\alpha \in (0, +\infty)$ (see [16]).

In [16, Lemma 2.1] it is proved that if $f \in \mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$, $\theta \in (0, 1)$, then for any $i, j \in I_0$, we have both $D_i f \in \mathcal{C}_d^{\theta+1}(\mathbb{R}^n)$ and $D_{ij}^2 f \in \mathcal{C}_d^\theta(\mathbb{R}^n)$; moreover there exists a constant C , independent of f , such that

$$\|D_i f\|_{1+\theta,d} + \|D_{ij}^2 f\|_{\theta,d} \leq C \|f\|_{2+\theta,d}, \quad i, j \in I_0. \quad (2.5)$$

Let $f \in \mathcal{C}_d^\gamma(\mathbb{R}^n)$, $\gamma \in (2, 3)$. For any $x \in \mathbb{R}^n$, we will consider $D_{E_0} f(x) \in \mathbb{R}^n$, the gradient of f at $x \in \mathbb{R}^n$ in the directions of $E_0(\mathbb{R}^n)$, i.e.,

$$D_{E_0} f(x) = (D_1 f(x), \dots, D_{\bar{p}} f(x), 0, \dots, 0) \quad (2.6)$$

and, similarly, the $n \times n$ Hessian matrix $D_{E_0}^2 f(x)$ in the directions of $E_0(\mathbb{R}^n)$, i.e., $(D_{E_0}^2 f(x))_{ij} = D_{ij}^2 f(x)$, if both i and $j \in I_0$; $(D_{E_0}^2 f(x))_{ij} = 0$ otherwise.

We finish the section with an equivalent definition of $\mathcal{C}_d^\gamma(\mathbb{R}^n)$ involving the quasi-norm $\|\cdot\|$. Let $f \in C_b(\mathbb{R}^n)$. We introduce, for any $x, v \in \mathbb{R}^n$,

$$\Delta_v^3 f(x) = f(x) - 3f(x+v) + 3f(x+2v) - f(x+3v). \quad (2.7)$$

Lemma 2.1. *Let $\gamma \in (0, 3)$ non-integer. Let $f \in C_b(\mathbb{R}^n)$. Then $f \in \mathcal{C}_d^\gamma(\mathbb{R}^n)$ if and only if*

$$[f]_{\gamma,d,3} = \sup_{x,v \in \mathbb{R}^n, v \neq 0, \|v\| \leq 1} |\Delta_v^3 f(x)| \|v\|^{-\gamma} < +\infty.$$

Moreover $\|\cdot\|_0 + [\cdot]_{\gamma,d,3}$ is equivalent to the norm $\|\cdot\|_{\gamma,d}$.

Proof. We use the following Triebel result (see [30, Section 2.7.2]). Let $g \in C_b(\mathbb{R}^n)$. Then g belongs to $\mathcal{C}_b^\gamma(\mathbb{R}^n)$, $\gamma \in (0, 3)$ non-integer, if and only if

$$[g]_{\gamma,3} = \sup_{x \in \mathbb{R}^n, |v| \leq 1, v \neq 0} |v|^{-\gamma} |\Delta_v^3 g(x)| < \infty. \quad (2.8)$$

Moreover in $\mathcal{C}_b^\gamma(\mathbb{R}^n)$ the norm $\|\cdot\|_\gamma$ is equivalent to $\|\cdot\|_0 + [\cdot]_{\gamma,3}$.

\implies Let $f \in \mathcal{C}_d^\gamma(\mathbb{R}^n)$ and fix $v \in \mathbb{R}^n$. We set $v = v_0 + v_1$, where $v_0 = E_0 v$ and $v_1 = \sum_{h=1}^k E_h v = v - E_0 v$, see (2.2). We get, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} |\Delta_v^3 f(x)| &\leq |f(x) - f(x+v_1)| \\ &\quad + |f(x+v_1) - 3f(x+v_1+v_0) + 3f(x+v_1+2v_0) - f(x+v_1+3v_0)| \\ &\quad + 3|f(x+2v_1+2v_0) - f(x+v_1+2v_0)| + |f(x+v_1+3v_0) - f(x+3v_1+3v_0)| \\ &\leq \|f\|_{\gamma,d} (4 \sum_{h=1}^k |E_h v|^{\frac{\gamma}{2h+1}} + \sum_{h=1}^k |E_h(2v)|^{\frac{\gamma}{2h+1}} + |v_0|^\gamma) \leq C \|f\|_{\gamma,d} \|v\|^\gamma. \end{aligned}$$

\Leftarrow Let $f \in C_b(\mathbb{R}^n)$ and take $v_h \in E_h(\mathbb{R}^n)$, with $0 \leq h \leq k$. By assumption, we know that $|\Delta_{v_h}^3 f(x)| \leq [f]_{\gamma,d,3} |v_h|^{\gamma/(2h+1)}$, for any $x \in \mathbb{R}^n$. It follows that $f(x+\cdot) \in C_b^{\gamma/(2h+1)}(E_h(\mathbb{R}^n))$ and there exists $C > 0$ independent of f and x such that $\|f(x+\cdot)\|_{C_b^{\gamma/(2h+1)}(E_h(\mathbb{R}^n))} \leq C (\|f\|_0 + [f]_{\gamma,d,3})$, $0 \leq h \leq k$. Thus $f \in \mathcal{C}_d^\gamma(\mathbb{R}^n)$. The proof is complete. \blacksquare

3 Estimates on the diffusion semigroup associated to \mathcal{A}

In this section we consider the diffusion semigroup P_t associated to the operator \mathcal{A} (compare with (1.7)). We obtain L^∞ -estimates on the first, second and third spatial partial derivatives of $P_t f$, in terms of the Hölder-norm of f . These estimates will lead in the next section to Schauder estimates for (1.2) and (1.3).

Let $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$ be a complete stochastic basis (satisfying the usual assumptions; see, for instance, [20]). Let W_t , $t \geq 0$, be a standard n -dimensional Wiener process defined and adapted on the stochastic basis. Let X_t^x be the unique (strong) solution to the SDE

$$X_t^x = x + \int_0^t A X_s^x ds + \int_0^t F(X_s^x) ds + Q^{1/2} W_t, \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad (3.1)$$

\mathbb{P} -a.s., where the matrix A is the same as in (1.1) and $Q^{1/2}$ is the unique $n \times n$ symmetric non-negative definite square root of Q . The *diffusion semigroup* P_t associated to \mathcal{A} is the family of linear contractions $P_t : B_b(\mathbb{R}^n) \rightarrow B_b(\mathbb{R}^n)$, $t \geq 0$, defined by

$$P_t g(x) := \mathbb{E}[g(X_t^x)], \quad t \geq 0, \quad g \in B_b(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \quad (3.2)$$

where the expectation is taken with respect to \mathbb{P} . Introducing the Ornstein-Uhlenbeck process Z_t^x , which solves (3.1) when $F = 0$, i.e.,

$$Z_t^x = e^{tA} x + Z_t^0, \quad \text{where } Z_t^0 = \int_0^t e^{(t-s)A} Q^{1/2} dW_s, \quad (3.3)$$

we have: $X_t^x = Z_t^x + \int_0^t e^{(t-s)A} F(X_s^x) ds$.

Let us recall an application of the Girsanov theorem which will be used in the proof of Theorem 3.3 (see also [25]). Fix $t > 0$, $x \in \mathbb{R}^n$, and define $Q^{-1/2} = \begin{pmatrix} Q_0^{-1/2} & 0 \\ 0 & 0 \end{pmatrix}$; then consider the stochastic process

$$L_s^x := W_s - \int_0^s (Q^{-1/2} F)(Z_r^x) dr = W_s - \int_0^s G(Z_r^x) dr, \quad s \in [0, t], \quad (3.4)$$

where we have set $G := Q^{-1/2} F$. By the Girsanov theorem, the process L_s^x is a Wiener process on $(\Omega, (\mathcal{F}_s)_{s \leq t}, \mathcal{F}_t, \mathbb{Q})$, where \mathbb{Q} is a probability measure on (Ω, \mathcal{F}_t) having density $\Phi(t, x)$ with respect to \mathbb{P} , i.e.,

$$\mathbb{Q}(A) := \mathbb{E}[1_A \Phi(t, x)], \quad \text{where } \Phi(t, x) = \exp \left(\int_0^t \langle G(Z_s^x), dW_s \rangle - \frac{1}{2} \int_0^t |G(Z_s^x)|^2 ds \right),$$

for any $A \in \mathcal{F}_t$. The processes $Z^x = (Z_s^x)$ and $X^x = (X_s^x)$, $s \in [0, t]$, satisfy the same equation (3.1) in $(\Omega, \mathcal{F}_t, \mathbb{Q}, (L_s^x))$ and $(\Omega, \mathcal{F}_t, \mathbb{P}, (W_s))$ respectively. Therefore, by uniqueness, the laws of the processes Z^x and X^x on $C([0, t]; \mathbb{R}^n)$ are the same (under the probability measures \mathbb{Q} and \mathbb{P} respectively). This implies that

$$P_t f(x) = \mathbb{E}[f(X_t^x)] = \mathbb{E}[f(Z_t^x) \Phi(t, x)], \quad f \in B_b(\mathbb{R}^n). \quad (3.5)$$

The next theorem is proved in [25]. It provides probabilistic formulae and preliminary uniform estimates for the spatial partial derivatives of $P_t f$ up to the third order (the formula for the first derivatives was obtained in [9]). The proof uses Malliavin Calculus. Related formulae for the spatial derivatives of degenerate diffusion semigroups are given in [3, 12].

Theorem 3.1. *Under Hypothesis 1.1, the following statements hold:*

(i) *For any $t > 0$ and $f \in B_b(\mathbb{R}^n)$, we have that $P_t f(\cdot)$ is three times differentiable on \mathbb{R}^n with all bounded derivatives up to the third order.*

(ii) *There exist random variables $J_i^1(t, x)$, $J_{ij}^2(t, x)$ and $J_{ijr}^3(t, x)$, $t > 0$, $x \in \mathbb{R}^n$, $i, j, r \in \{1, \dots, n\}$, which belong to $L^q(\Omega)$, for any $q \geq 1$, and such that*

$$\begin{aligned} D_i(P_t g)(x) &= D_i P_t g(x) = \mathbb{E}[g(X_t^x) J_i^1(t, x)], & D_{ij}^2 P_t g(x) &= \mathbb{E}[g(X_t^x) J_{ij}^2(t, x)], \\ D_{ijr}^3 P_t g(x) &= \mathbb{E}[g(X_t^x) J_{ijr}^3(t, x)], & g &\in C_b(\mathbb{R}^n). \end{aligned} \quad (3.6)$$

(iii) *For any $t > 0$, $q \geq 1$, we have the following estimates:*

$$\begin{aligned} (a) \quad &\mathbb{E}|J_i^1(t, x)|^q \leq c_q(t) |Q_t^{-1/2} e^{tA} e_i|^q, \\ (b) \quad &\mathbb{E}|J_{ij}^2(t, x)|^q \leq c_q(t) |Q_t^{-1/2} e^{tA} e_i|^q |Q_t^{-1/2} e^{tA} e_j|^q, \\ (c) \quad &\mathbb{E}|J_{ijr}^3(t, x)|^q \leq c_q(t) |Q_t^{-1/2} e^{tA} e_i|^q |Q_t^{-1/2} e^{tA} e_j|^q |Q_t^{-1/2} e^{tA} e_r|^q, \quad x \in \mathbb{R}^n, \end{aligned} \quad (3.7)$$

where $c_q(t)$ is a continuous and increasing function on $[0, \infty)$; $c_q(t) = c(q, t, \|DF\|_0, \|D^2F\|_0, \|D^3F\|_0, \tilde{p}, \nu_1, A, n)$, where the integer \tilde{p} is introduced in (1.4).

It is worth noticing that the quantity $|Q_t^{-1/2} e^{tA} e_i|^2$, corresponding to $q = 2$, has a well-known control-theoretic interpretation; see, for instance, [31].

Moreover, the following estimated are known, see [28] and [16, formula (3.4)],

$$|Q_t^{-1/2} e^{tA} e_i| \leq \frac{c}{t^{h+1/2}}, \quad e_i \in E_h(\mathbb{R}^n), \quad 0 \leq h \leq k, \quad t \in (0, 1]. \quad (3.8)$$

where $c = c(\tilde{p}, \nu_1, \nu_2, A, n) > 0$ and the integer k is defined in (1.5). Estimates (3.8) can be also deduced by pure control theoretic arguments. To this purpose one has to use [31, Proposition I.1.3] together with [1].

Corollary 3.2. *There exists $c = c(\tilde{p}, \nu_1, \nu_2, A, n, \|DF\|_0, \|D^2F\|_0, \|D^3F\|_0) > 0$ such that the following estimates hold, for any $t > 0$, $g \in B_b(\mathbb{R}^n)$, indices $i \in I_h$, $j \in I_{h'}$ and $r \in I_{h''}$, where $h, h', h'' \in \{0, \dots, k\}$,*

$$\begin{aligned} \|D_i P_t g\|_0 &\leq c \left(\frac{1}{t^{h+1/2}} + 1 \right) \|g\|_0; & \|D_{ij}^2 P_t g\|_0 &\leq c \left(\frac{1}{t^{h+h'+1}} + 1 \right) \|g\|_0; \\ \|D_{ijr}^3 P_t g\|_0 &\leq c \left(\frac{1}{t^{h+h'+h''+3/2}} + 1 \right) \|g\|_0. \end{aligned} \quad (3.9)$$

Proof. It is enough to prove the estimates when $g \in C_b(\mathbb{R}^n)$ (see, for instance, [25, Remark 3.5]). Using Theorem 3.1 and formula (3.8), we first establish the estimates assuming in addition that $0 < t < 1$. We have, for any $x \in \mathbb{R}^n$, $t \in (0, 1)$,

$$|D_i P_t g(x)| \leq \|g\|_0 \mathbb{E}|J_i^1(t, x)| \leq c_1 |Q_t^{-1/2} e^{tA} e_i| \|g\|_0 \leq \frac{c}{t^{h+1/2}} \|g\|_0.$$

In a similar way, we get the second and third estimates, for $t < 1$.

When $t \geq 1$, by the semigroup and the contraction property of P_t , we have:

$$\|D_i P_t g\|_0 = \|D_i P_{1/2}(P_{t-\frac{1}{2}} g)\|_0 \leq c 2^{h+1/2} \|P_{\frac{2t-1}{2}} g\|_0 \leq c 2^{h+1/2} \|g\|_0,$$

for any $i \in I_h$. Thus the required estimate of $D_i P_t g$ is proved for any $t > 0$. Similarly, we get the other estimates for any $t > 0$. \blacksquare

The main result of the section is the following theorem. Its proof also allows to complete the final part of the proof of [16, Theorem 3.4]. We set $t \wedge 1 = \min(t, 1)$.

Theorem 3.3. *Fix any $\gamma \in (0, 3)$ non-integer. There exists $c = c(\gamma, \tilde{p}, \nu_1, \nu_2, A, n, \|DF\|_0, \|D^2F\|_0, \|D^3F\|_0) > 0$, such that, for any $f \in C_d^\gamma(\mathbb{R}^n)$, $t > 0$, for any indices $i \in I_h$, $j \in I_{h'}$ and $r \in I_{h''}$, where $h, h', h'' \in \{0, \dots, k\}$, it holds*

$$\begin{aligned} (i) \quad & \|D_i P_t f\|_0 \leq c \left(\frac{1}{(t \wedge 1)^{\frac{1-\gamma}{2}+h}} + 1 \right) \|f\|_{\gamma,d}; & (ii) \quad & \|D_{ij}^2 P_t f\|_0 \leq c \left(\frac{1}{(t \wedge 1)^{h+h'+\frac{2-\gamma}{2}}} + 1 \right) \|f\|_{\gamma,d}; \\ (iii) \quad & \|D_{ijr}^3 P_t f\|_0 \leq c \left(\frac{1}{t^{h+h'+h''+\frac{3-\gamma}{2}}} + 1 \right) \|f\|_{\gamma,d}; & (iv) \quad & \|P_t f\|_{\gamma,d} \leq c \|f\|_{\gamma,d}. \end{aligned} \quad (3.10)$$

Remark 3.4. Estimates (i)-(iv) will be used to get elliptic and parabolic Schauder estimates for \mathcal{A} . However, we stress that to prove *elliptic Schauder estimates* we only need a special case of (3.10). More precisely, we need the following estimates, for any $\theta \in (0, 1)$, $f \in C_d^\theta(\mathbb{R}^n)$, $t > 0$, for any indices $i, j \in I_0$, $r \in I_h$, with $h \in \{0, \dots, k\}$,

$$\begin{aligned} (a) \quad & \|D_r P_t f\|_0 \leq c \left(\frac{1}{t^{\frac{1-\theta}{2}+h}} + 1 \right) \|f\|_{\theta,d}; & (b) \quad & \|D_{ij}^2 P_t f\|_0 \leq c \left(\frac{1}{t^{\frac{2-\theta}{2}}} + 1 \right) \|f\|_{\theta,d}; \\ (c) \quad & \|D_{ijr}^3 P_t f\|_0 \leq c \left(\frac{1}{t^{\frac{3-\theta}{2}+h}} + 1 \right) \|f\|_{\theta,d}; & (d) \quad & \|P_t f\|_{\theta,d} \leq c \|f\|_{\theta,d}. \end{aligned} \quad (3.11)$$

These estimates are simpler to obtain than the general ones in which $\gamma \in (0, 3)$. On the other hand, note that the estimates (iv) in (3.10) with $\gamma \in (2, 3)$ are a particular case of the parabolic Schauder estimates corresponding to $H = 0$ in (1.3) (see Theorem 4.3). Estimates (iv) with $\gamma \in (2, 3)$ will be deduced by (iii). \blacksquare

In order to prove the main result we need three preliminary lemmas. To state the first one, we introduce the deterministic process Y_t^x ,

$$Y_t^x = e^{tA}x + \int_0^t e^{(t-s)A} F(Y_s^x) ds, \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad \text{which solves} \quad \begin{cases} \dot{Y}_t = AY_t + F(Y_t), \\ Y_0 = x, \end{cases} \quad (3.12)$$

Lemma 3.5. *For any $q > 0$, there exists $C = C(q, \tilde{p}, \nu_1, \nu_2, n, A, \|DF\|_0) > 0$, such that*

$$\sup_{x \in \mathbb{R}^n} \mathbb{E}[(d(X_t^x, Y_t^x))^q] = \sup_{x \in \mathbb{R}^n} \mathbb{E} \|X_t^x - Y_t^x\|^q \leq C t^{\frac{q}{2}}, \quad 0 \leq t \leq 1. \quad (3.13)$$

Proof. Note that (3.13) is equivalent to the following assertion: for any $q > 0$, $0 \leq h \leq k$, there exists $C_1 > 0$ such that

$$\sup_{x \in \mathbb{R}^n} \mathbb{E} |E_h(X_t^x - Y_t^x)|^q \leq C_1 t^{\frac{q}{2}(2h+1)}, \quad 0 \leq t \leq 1, \quad (3.14)$$

see (2.3). Let us prove (3.14). Since there exists $c > 0$, such that $|x| \leq c \sum_{h=0}^k |E_h x|$, for any $x \in \mathbb{R}^n$, we get

$$\begin{aligned} |E_h(X_t^x - Y_t^x)| & \leq \left| \int_0^t (E_h e^{(t-r)A} E_0) [F(X_r^x) - F(Y_r^x)] dr \right| + |E_h Z_t^0| \\ & \leq c \|DF\|_0 \sum_{j=0}^k \int_0^t \|E_h e^{(t-r)A} E_0\|_L |E_j(X_r^x - Y_r^x)| dr + |E_h Z_t^0|, \end{aligned}$$

\mathbb{P} -a.s.. Using the following estimate, see [16, Lemma 3.1],

$$\|E_h e^{sA} E_0\|_L \leq c'' s^h, \quad s \in [0, 1], \quad 0 \leq h \leq k, \quad \text{where } c'' = c''(A) > 0, \quad (3.15)$$

we arrive at

$$\begin{aligned} & |E_h(X_t^x - Y_t^x)| \leq |E_h Z_t^0| \\ & + C \int_0^t (t-r)^h |E_h(X_r^x - Y_r^x)| dr + C \sum_{j=0, j \neq h}^k \int_0^t (t-r)^h |E_j(X_r^x - Y_r^x)| dr. \end{aligned} \quad (3.16)$$

\mathbb{P} - a.s.. Now we use that $|E_j(X_r^x - Y_r^x)| \leq |X_r^x - Y_r^x|$, for any $0 \leq j \leq k$. In order to estimate $|X_r^x - Y_r^x|$, note that

$$|X_t^x - Y_t^x| \leq C' \int_0^t |X_s^x - Y_s^x| ds + |Z_t^0|.$$

An application of the Gronwall lemma gives, \mathbb{P} -a.s.,

$$|X_t^x - Y_t^x| \leq |Z_t^0| + C' \int_0^t |Z_s^0| e^{(t-s)C'} ds \leq |Z_t^0| + C_1 \int_0^t |Z_s^0| ds, \quad 0 \leq t \leq 1. \quad (3.17)$$

Using estimate (3.17) in (3.16) we get

$$\begin{aligned} & |E_h(X_t^x - Y_t^x)| \\ & \leq |E_h Z_t^0| + C \int_0^t (t-r)^h |E_h(X_r^x - Y_r^x)| dr + C \sum_{j \neq h} \int_0^t (t-r)^h \left[|Z_r^0| + C_1 \int_0^r |Z_s^0| ds \right] dr, \end{aligned}$$

\mathbb{P} -a.s.. Let now $q \in \mathbb{Z}_+$ and recall that $0 \leq t \leq 1$. We have

$$\begin{aligned} & |E_h(X_t^x - Y_t^x)|^q \\ & \leq C_3 \left(|E_h Z_t^0|^q + \int_0^t |E_h(X_r^x - Y_r^x)|^q dr + \sum_{j \neq h} \int_0^t (t-r)^{hq} \left[|Z_r^0|^q + \int_0^r |Z_s^0|^q ds \right] dr \right), \end{aligned}$$

\mathbb{P} -a.s.. Before applying the expectation in the last formula, we check that

$$\mathbb{E} |E_h Z_t^0|^q = \mathbb{E} \left| \int_0^t E_h e^{(t-s)A} Q^{1/2} dW_s \right|^q \leq c_{q,h} t^{q(2h+1)/2}, \quad 0 \leq t \leq 1, \quad q > 0, \quad 0 \leq h \leq k, \quad (3.18)$$

where E_h are the orthogonal projections introduced in (2.2). Denoting by $N(0, Q_t)$ the Gaussian measure on \mathbb{R}^n with mean 0 and covariance matrix Q_t given in (2.1), we have:

$$\begin{aligned} & \mathbb{E} |E_h Z_t^0|^q = \int_{\mathbb{R}^n} |E_h y|^q N(0, Q_t) dy \\ & = \int_{\mathbb{R}^n} |E_h Q_t^{1/2} z|^q N(0, I) dz \leq \|E_h Q_t^{1/2}\|_L^q \int_{\mathbb{R}^n} |z|^q N(0, I) dz \leq c t^{q(2h+1)/2}, \quad t \leq 1, \end{aligned} \quad (3.19)$$

where I is the $n \times n$ identity matrix. In the last inequality we have used that $\|E_h Q_t^{1/2}\|_L \leq c' t^{(2h+1)/2}$, $0 \leq t \leq 1$, $0 \leq h \leq k$, where $c' = c'(\tilde{p}, n, A, \nu_1, \nu_2)$ (see [16, formula (3.2)]).

By (3.18), we infer

$$\begin{aligned} \mathbb{E}|E_h(X_t^x - Y_t^x)|^q &\leq C_3 \left(\mathbb{E}|E_h Z_t^0|^q + \int_0^t \mathbb{E}|E_h(X_r^x - Y_r^x)|^q dr \right. \\ &\quad \left. + \sum_{j \neq h} \int_0^t (t-r)^{hq} \left[\mathbb{E}|Z_r^0|^q + \int_0^r \mathbb{E}|Z_s^0|^q ds \right] dr \right) \\ &\leq C_4 \left(t^{\frac{q(2h+1)}{2}} + \int_0^t \mathbb{E}|E_h(X_r^x - Y_r^x)|^q dr + \sum_{j \neq h} \int_0^t (t-r)^{hq} \left[r^{q/2} + \frac{r^{1+q/2}}{1+q/2} \right] dr \right). \end{aligned}$$

Using that $\int_0^t (t-s)^p s^r ds = \frac{t^{p+1}}{(r+p+1)(r+p)\dots(r+1)}$, for $p \in \mathbb{Z}_+$, $r > 0$, we get

$$\begin{aligned} \mathbb{E}|E_h(X_t^x - Y_t^x)|^q &\leq C_5 \left(t^{\frac{q(2h+1)}{2}} + \int_0^t \mathbb{E}|E_h(X_r^x - Y_r^x)|^q dr + 2 \sum_{j \neq h} \int_0^t (t-r)^{hq} r^{q/2} dr \right) \\ &\leq C_6 \left(t^{\frac{q(2h+1)}{2}} + \int_0^t \mathbb{E}|E_h(X_r^x - Y_r^x)|^q dr + t^{hq+1+q/2} \right) \\ &\leq 2C_6 \left(t^{\frac{q(2h+1)}{2}} + \int_0^t \mathbb{E}|E_h(X_r^x - Y_r^x)|^q dr \right), \quad t \leq 1. \end{aligned}$$

Applying the Gronwall lemma, we get

$$\mathbb{E}|E_h(X_t^x - Y_t^x)|^q \leq C_7 t^{\frac{q(2h+1)}{2}}, \quad 0 \leq t \leq 1.$$

Now if $q \in \mathbb{R}_+$, $q > 0$, we consider an integer $m \geq q$. By the Jensen inequality,

$$\left(\mathbb{E}|E_h(X_t^x - Y_t^x)|^q \right)^{m/q} \leq \mathbb{E}|E_h(X_t^x - Y_t^x)|^m \leq C t^{\frac{m(2h+1)}{2}}, \quad t \leq 1.$$

This implies that $\mathbb{E}|E_h(X_t^x - Y_t^x)|^q \leq C^{q/m} t^{\frac{q(2h+1)}{2}}$. The assertion is proved. \blacksquare

Lemma 3.6. *For any ω , \mathbb{P} -a.s., $t \in [0, 1]$, the mapping $x \mapsto X_t^x(\omega) \in \mathbb{R}^n$ is differentiable up to the third order on \mathbb{R}^n . Moreover, for any $i, j, r \in \{0, \dots, n\}$, $x \in \mathbb{R}^n$, there exist continuous adapted stochastic processes $(\eta_i(t, x))$, $(\eta_{ij}(t, x))$ and $(\eta_{ijr}(t, x))$ with values in \mathbb{R}^n and $C = C(\|DF\|_0, \|D^2F\|_0, \|D^3F\|_0, \|A\|_L) > 0$ such that*

$$\begin{aligned} \eta_i(t, x) &= D_i X_t^x = \lim_{h \rightarrow 0} (X_t^{x+he_i} - X_t^x) h^{-1}, \quad \eta_{ij}(t, x) = D_{ij}^2 X_t^x, \quad \eta_{ijr}(t, x) = D_{ijr}^3 X_t^x \\ \text{and } |\eta_i(t, x)| + |\eta_{ij}(t, x)| + |\eta_{ijr}(t, x)| &\leq C, \quad \text{for any } t \in [0, 1], x \in \mathbb{R}^n, \omega \in \Omega, \mathbb{P}\text{-a.s.} \end{aligned}$$

Proof. The proof is straightforward. We include it for the sake of completeness. Fix $\omega \in \Omega$, \mathbb{P} -a.s., and introduce the Banach space $E = C([0, 1]; \mathbb{R}^n)$. Define the map $\mathcal{F} : \mathbb{R}^n \times E \rightarrow E$,

$$\mathcal{F}(x, u)(t) := u(t) - x - \int_0^t (Au(r) + F(u(r))) dr - \sqrt{Q}W_t(\omega), \quad t \in [0, 1], u \in E, x \in \mathbb{R}^n.$$

Applying the implicit function theorem, we find that the mapping: $x \mapsto X_{(\cdot)}^x(\omega)$ from \mathbb{R}^n into E is three times Fréchet-differentiable. Denote by $\eta_i(t, x)$, $\eta_{ij}(t, x)$ and $\eta_{ijr}(t, x)$ $t \in [0, 1]$, respectively the first (directional) derivative at $x \in \mathbb{R}^n$ in the direction e_i , the second derivative at x in the directions e_i and e_j , and the third derivative at x in the

directions e_i, e_j and e_r , where $i, j, r = 1, \dots, n$. Note that $\eta_i(t, x)$, $\eta_{ij}(t, x)$ and $\eta_{ijr}(t, x)$ solve, \mathbb{P} -a.s., the variation equations

$$\begin{aligned}\eta_i(t, x) &= e_i + \int_0^t (A\eta_i(s, x) + DF(X_s^x)[\eta_i(s, x)]) ds; \\ \eta_{ij}(t, x) &= \int_0^t (A\eta_{ij}(s, x) + D^2F(X_s^x)[\eta_i(s, x)][\eta_j(s, x)] + DF(X_s^x)[\eta_{ij}(s, x)]) ds; \\ \eta_{ijr}(t, x) &= \int_0^t (A\eta_{ijr}(s, x) + D^3F(X_s^x)[\eta_i(s, x)][\eta_j(s, x)][\eta_r(s, x)]) ds \\ &\quad + \int_0^t (D^2F(X_s^x)[\eta_{ir}(s, x)][\eta_j(s, x)] + D^2F(X_s^x)[\eta_i(s, x)][\eta_{jr}(s, x)] + DF(X_s^x)[\eta_{ijr}(s, x)]) ds,\end{aligned}$$

$t \in [0, 1]$. It follows easily that $(\eta_i(\cdot, x))$, $(\eta_{ij}(\cdot, x))$ and $(\eta_{ijr}(\cdot, x))$ are continuous adapted stochastic processes. An application of the Gronwall lemma gives the final assertion. \blacksquare

Lemma 3.7. *Let $f \in C_d^\gamma(\mathbb{R}^n)$, $\gamma \in (2, 3)$, and $i, j, r \in \{1, \dots, n\}$. Consider the following random variables depending on $t \in (0, 1)$ and $x \in \mathbb{R}^n$ (see (2.6) and (3.6))*

$$\Lambda(t, x) = \langle D_{E_0} f(Y_t^x), E_0(X_t^x - Y_t^x) \rangle + \frac{1}{2} \langle D_{E_0}^2 f(Y_t^x) [E_0(X_t^x - Y_t^x)], E_0(X_t^x - Y_t^x) \rangle.$$

Then the functions: $\phi_i(x, t) = \mathbb{E}[\Lambda(t, x) J_i^1(t, x)]$, $\phi_{ij}(x, t) = \mathbb{E}[\Lambda(t, x) J_{ij}^2(t, x)]$, $\phi_{ijr}(x, t) = \mathbb{E}[\Lambda(t, x) J_{ijr}^3(t, x)]$, $x \in \mathbb{R}^n$, $t \in (0, 1)$, are continuous and bounded on $\mathbb{R}^n \times (0, 1)$.

Proof. Let us treat ϕ_i . We introduce the deterministic functions $K : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}$,

$$K(x, z, t) = \langle D_{E_0} f(Y_t^z), E_0(x - Y_t^z) \rangle + \frac{1}{2} \langle D_{E_0}^2 f(Y_t^z) [E_0(x - Y_t^z)], E_0(x - Y_t^z) \rangle \quad (3.20)$$

and $g_i : \mathbb{R}^n \times \mathbb{R}^n \times (0, 1] \rightarrow \mathbb{R}$,

$$g_i(x, z, t) = \mathbb{E}[K(X_t^x, z, t) J_i^1(t, x)], \quad x, z \in \mathbb{R}^n, t \in (0, 1].$$

Note that $\phi_i(x, t) = g_i(x, x, t)$, $x \in \mathbb{R}^n$, $t \in (0, 1)$. We first prove that

$$g_i(x, z, t) = D_i(\mathbb{E}[K(X_t^{(\cdot)}, z, t)])(x) = \mathbb{E}[\langle D_x K(X_t^x, z, t), \eta_i(t, x) \rangle], \quad (3.21)$$

$x, z \in \mathbb{R}^n$, $t \in (0, 1)$ (here $D_{x_i} = D_i$ denotes the partial derivative with respect to e_i and D_x denotes the gradient in the x -variable; η_i is introduced in Lemma 3.6). To this purpose, remark that it holds

$$|K(x, z, t)| + |D_{x_i} K(x, z, t)| + |D_{x_i x_j}^2 K(x, z, t)| + |D_{x_i x_j x_r}^3 K(x, z, t)| \leq 8 \|f\|_{\gamma, d} (1 + |E_0(x - Y_t^z)|^2), \quad (3.22)$$

$t \in [0, 1]$, $x, z \in \mathbb{R}^n$, $i, j, r \in \{1, \dots, n\}$. Moreover, an application of the Gronwall lemma shows that

$$|X_t^x| \leq e^{(\|A\|_L + \|DF\|_0)t} (|x| + |F(0)| + \|\sqrt{Q}\|_L \sup_{s \leq 1} |W_s|), \quad t \in [0, 1], x \in \mathbb{R}^n, \quad (3.23)$$

\mathbb{P} -a.s.. By (3.22) and (3.23), using Lemma 3.6, we get the existence of the partial derivatives

$$D_{x_i}(\mathbb{E}[K(X_t^{(\cdot)}, z, t)])(x) = \mathbb{E}[\langle D_x K(X_t^x, z, t), \eta_i(t, x) \rangle], \quad x, z \in \mathbb{R}^n, t \in (0, 1), 1 \leq i \leq n.$$

To obtain (3.21), we consider test functions $\varphi_m \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi_m \leq 1$, $m \in \mathbb{N}$, $\varphi_m(x) = 1$, when $|x| \leq m$, $\varphi_m(x) = 0$, when $|x| > m + 1$ and $|D\varphi_m(x)| \leq 1$, for $x \in \mathbb{R}^n$, $m \in \mathbb{N}$. By Theorem 3.1 and Lemma 3.6, we know that, for $x, z \in \mathbb{R}^n$, $t \in (0, 1)$, $m \in \mathbb{N}$,

$$\begin{aligned} D_{x_i}(\mathbb{E}[K(X_t^{(\cdot)}, z, t) \varphi_m(X_t^{(\cdot)})])(x) &= \mathbb{E}[K(X_t^x, z, t) \varphi_m(X_t^x) J_i^1(t, x)] \\ &= \mathbb{E}[\langle D_x K(X_t^x, z, t), \eta_i(t, x) \rangle \varphi_m(X_t^x) + K(X_t^x, z, t) \langle D\varphi_m(X_t^x), \eta_i(t, x) \rangle]. \end{aligned}$$

Passing to the limit as $m \rightarrow \infty$, we get (3.21), by the dominated convergence theorem. By (3.21), (3.22) and Lemma 3.6, we have (setting $z = x$)

$$\begin{aligned} |\phi_i(x, t)| &= |g_i(x, x, t)| \leq \mathbb{E}[|D_{x_i} K(X_t^x, x, t)| |\eta_i(t, x)|] \\ &\leq 8\|\eta_i\|_{L^\infty} \|f\|_{\gamma, d} (1 + \mathbb{E}[|E_0(X_t^x - Y_t^x)|^2]), \end{aligned}$$

for any $x \in \mathbb{R}^n$, $t \in (0, 1)$. Applying Lemma 3.5, we get

$$\sup_{x \in \mathbb{R}^n, t \in (0, 1)} |\phi_i(x, t)| \leq 8c\|\eta_i\|_{L^\infty} \|f\|_{\gamma, d} \sup_{t \in (0, 1)} |1 + t| \leq 16c\|\eta_i\|_{L^\infty} \|f\|_{\gamma, d}.$$

To treat ϕ_{ij} and ϕ_{ijr} we proceed similarly. Concerning ϕ_{ij} we introduce

$$\begin{aligned} g_{ij}(x, z, t) &= \mathbb{E}[K(X_t^x, z, t) J_{ij}^2(t, x)] = D_{ij}^2(\mathbb{E}[K(X_t^{(\cdot)}, z, t)])(x) \\ &= \mathbb{E}[\langle D_x^2 K(X_t^x, z, t) [\eta_j(t, x)], \eta_i(t, x) \rangle + \langle D_x K(X_t^x, z, t), \eta_{ij}(t, x) \rangle]. \end{aligned}$$

Since $\phi_{ij}(x, t) = g_{ij}(x, x, t)$, we obtain the assertion for ϕ_{ij} , using (3.22), Lemmas 3.6 and 3.5, as before. To treat ϕ_{ijr} we introduce $g_{ijr}(x, z, t) = \mathbb{E}[K(X_t^x, z, t) J_{ijr}^3(t, x)]$. Note that

$$\begin{aligned} g_{ijr}(x, z, t) &= \mathbb{E}[\langle D_x^3 K(X_t^x, z, t) [\eta_r(t, x)] [\eta_j(t, x)], \eta_i(t, x) \rangle + \langle D_x^2 K(X_t^x, z, t) [\eta_{jr}(t, x)], \eta_i(t, x) \rangle \\ &\quad + \langle D_x^2 K(X_t^x, z, t) [\eta_j(t, x)], \eta_{ir}(t, x) \rangle + \langle D_x^2 K(X_t^x, z, t) [\eta_r(t, x)], \eta_{ij}(t, x) \rangle \\ &\quad + \langle D_x K(X_t^x, z, t), \eta_{ijr}(t, x) \rangle]. \end{aligned}$$

Since $\phi_{ijr}(x, t) = g_{ijr}(x, x, t)$, we get the assertion for ϕ_{ijr} proceeding as for ϕ_i and ϕ_{ij} . The proof is complete. \blacksquare

Proof of Theorem 3.3. Thanks to Corollary 3.2, *it is enough to prove all the estimates for $0 < t < 1$* . Indeed, concerning (iv) in (3.10), we have, for $t \geq 1$,

$$\begin{aligned} \|P_t f\|_{\gamma, d} &= \sum_{m=0}^k \sup_{z \in \mathbb{R}^n} \|(P_t f)(z + \cdot)\|_{C_b^{\gamma/(2m+1)}(E_m(\mathbb{R}^n))} \\ &\leq c' \sum_{m=0}^k (\|f\|_0 + \sup_{z \in \mathbb{R}^n} \|(P_t f)(z + \cdot)\|_{C_b^3(E_m(\mathbb{R}^n))}) \leq c_2 \|f\|_0, \quad f \in \mathcal{C}_d^\gamma(\mathbb{R}^n). \end{aligned}$$

We will show the estimates for $\gamma \in (2, 3)$.

Indeed, the cases of $\gamma \in (0, 1)$ and $\gamma \in (1, 2)$ can be similarly treated and are even simpler. Alternatively, once we have proved the estimates for $\gamma \in (2, 3)$, the remaining estimates can be obtained by an interpolation argument. Let us briefly explain such method which has been also used in the proof of [16, Theorem 3.4]. We assume that (i)-(iv) hold for $\gamma = 5/2$

and show that they hold also for a fixed $\gamma' \in (0, 2)$, $\gamma' \neq 1$. By [16, Theorem 2.2], we know in particular that

$$(C_b(\mathbb{R}^n), \mathcal{C}_d^{5/2}(\mathbb{R}^n))_{2\gamma'/5, \infty} = \mathcal{C}_d^{\gamma'}(\mathbb{R}^n). \quad (3.24)$$

To be precise, (3.24) is proved in [16] when $C_b(\mathbb{R}^n)$ denotes the Banach space of all real continuous and bounded functions defined on \mathbb{R}^n . However, the same proof of [16] works also when we consider $C_b(\mathbb{R}^n)$ as the space of all real uniformly continuous and bounded functions. Concerning estimate (iv) in (3.10), by (3.24) and [18, Proposition 1.2.6] we get

$$\|P_t\|_{L(\mathcal{C}_d^{\gamma'}(\mathbb{R}^n), \mathcal{C}_d^{\gamma'}(\mathbb{R}^n))} \leq (\|P_t\|_{L(C_b(\mathbb{R}^n), C_b(\mathbb{R}^n))})^{1-\frac{2\gamma'}{5}} (\|P_t\|_{L(\mathcal{C}_d^{5/2}(\mathbb{R}^n), \mathcal{C}_d^{5/2}(\mathbb{R}^n))})^{\frac{2\gamma'}{5}} \leq C,$$

for $t \geq 0$. As for (iii), we fix $x \in \mathbb{R}^n$, $t \in (0, 1]$ and introduce the linear operator $T_{x,t} : \mathcal{C}_d^{\gamma'}(\mathbb{R}^n) \rightarrow \mathbb{R}$, $T_{x,t}f := D_{ijr}^3 P_t f(x)$, for any $f \in \mathcal{C}_d^{\gamma'}(\mathbb{R}^n)$. We have:

$$\|T_{x,t}\|_{L(\mathcal{C}_d^{\gamma'}(\mathbb{R}^n), \mathbb{R})} \leq (\|T_{x,t}\|_{L(C_b(\mathbb{R}^n), \mathbb{R})})^{1-\frac{2\gamma'}{5}} (\|T_{x,t}\|_{L(\mathcal{C}_d^{5/2}(\mathbb{R}^n), \mathbb{R})})^{\frac{2\gamma'}{5}} \leq ct^{-(\frac{3}{2}+h+h'+h'')+\frac{\gamma'}{2}},$$

$t \in (0, 1]$ (uniformly in $x \in \mathbb{R}^n$). In a similar way, one can prove (i) and (ii) for γ' .

- We prove the first estimate in (3.10), for $t \in (0, 1)$, $\gamma \in (2, 3)$ and $i \in I_h$.

We start from (3.6) and write

$$\begin{aligned} D_i P_t f(x) &= \Lambda_1(t, x) + \Lambda_2(t, x) \quad \text{where} \\ \Lambda_1(t, x) &= \mathbb{E}[\{f(X_t^x) - f(E_0 X_t^x + \sum_{h=1}^k E_h Y_t^x)\} J_i^1(t, x)]; \\ \Lambda_2(t, x) &= \mathbb{E}[f(E_0 X_t^x + \sum_{h=1}^k E_h Y_t^x) J_i^1(t, x)], \end{aligned} \quad (3.25)$$

where (Y_t^x) is defined in (3.12). Let us treat Λ_1 and Λ_2 separately. Using (3.7), (3.8) and Lemma 3.5, we have (since $0 < \gamma/(2m+1) < 1$ if $m = 1, \dots, k$)

$$\begin{aligned} |\Lambda_1(t, x)| &\leq c \|f\|_{\gamma, d} \mathbb{E} \left[\left\{ \sum_{m=1}^k |E_m(X_t^x - Y_t^x)|^{\frac{\gamma}{2m+1}} \right\} |J_i^1(t, x)| \right] \\ &\leq c' \|f\|_{\gamma, d} \left(\mathbb{E} \|X_t^x - Y_t^x\|^{2\gamma} \right)^{1/2} \left(\mathbb{E} |J_i^1(t, x)|^2 \right)^{1/2} \leq c_2 \|f\|_{\gamma, d} t^{\frac{\gamma}{2}} t^{-(h+1/2)} = c_2 \|f\|_{\gamma, d} t^{\frac{\gamma-1}{2}-h}, \end{aligned}$$

$t \in (0, 1)$, uniformly in $x \in \mathbb{R}^n$. Let us concentrate on the more difficult term Λ_2 . We write

$$\begin{aligned} \Lambda_2(t, x) &= \Lambda_{21}(t, x) + \Lambda_{22}(t, x), \quad \text{where} \\ \Lambda_{21}(t, x) &= \mathbb{E} \left[\left(f(E_0 X_t^x + \sum_{m=1}^k E_m Y_t^x) - f(Y_t^x) - \langle D_{E_0} f(Y_t^x), E_0(X_t^x - Y_t^x) \rangle \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \langle D_{E_0}^2 f(Y_t^x) [E_0(X_t^x - Y_t^x)], E_0(X_t^x - Y_t^x) \rangle \right) J_i^1(t, x) \right], \\ \Lambda_{22}(t, x) &= \mathbb{E} \left[\left(\langle D_{E_0} f(Y_t^x), E_0(X_t^x - Y_t^x) \rangle \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \langle D_{E_0}^2 f(Y_t^x) [E_0(X_t^x - Y_t^x)], E_0(X_t^x - Y_t^x) \rangle \right) J_i^1(t, x) \right], \end{aligned}$$

see (2.6). Note that, since (Y_t^x) is deterministic, $\mathbb{E}[f(Y_t^x) J_i^1(t, x)] = f(Y_t^x) \mathbb{E}[J_i^1(t, x)] = f(Y_t^x) D_i(P_t 1)(x) = 0$, for any $x \in \mathbb{R}^n$, $t > 0$.

To estimate Λ_{21} , remark that $f(x + \cdot) \in C_b^\gamma(E_0(\mathbb{R}^n))$, $\gamma \in (2, 3)$, uniformly in x . By the mean value theorem, we have:

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} |\Lambda_{21}(t, x)| &\leq \|f\|_{\gamma, d} \sup_{x \in \mathbb{R}^n} \{ \mathbb{E} [|E_0(X_t^x - Y_t^x)|^\gamma |J_i^1(t, x)|] \} \\ &\leq \|f\|_{\gamma, d} \sup_{x \in \mathbb{R}^n} \left(\mathbb{E} [|E_0(X_t^x - Y_t^x)|^{2\gamma}] \right)^{1/2} \cdot \sup_{x \in \mathbb{R}^n} \left(\mathbb{E} |J_i^1(t, x)|^2 \right)^{1/2} \leq c_3 \|f\|_{\gamma, d} t^{\frac{\gamma-1}{2}-h}, \end{aligned}$$

see also (3.14). Finally, using Lemma 3.7, we infer $\sup_{x \in \mathbb{R}^n, t \in (0,1)} |\Lambda_{22}(t, x)| = c_4 < \infty$. This proves the estimate.

- We prove (ii) and (iii) in (3.10), for $t \in (0, 1)$ and $\gamma \in (2, 3)$.

These estimates can be obtained proceeding as in the proof of the first estimate in (3.10). We only prove (ii). Let $i \in I_h$ and $j \in I_{h'}$. We write

$$D_{ij}^2 P_t f(x) = \tilde{\Lambda}_1(t, x) + \tilde{\Lambda}_{21}(t, x) + \tilde{\Lambda}_{22}(t, x), \quad \text{where}$$

$$\begin{aligned} \tilde{\Lambda}_1(t, x) &= \mathbb{E}[\{f(X_t^x) - f(E_0 X_t^x + \sum_{h=1}^k E_h Y_t^x)\} J_{ij}^2(t, x)], \\ \tilde{\Lambda}_{21}(t, x) &= \mathbb{E}\left[\left(f(E_0 X_t^x + \sum_{h=1}^k E_h Y_t^x) - f(Y_t^x) - \langle D_{E_0} f(Y_t^x), E_0(X_t^x - Y_t^x) \rangle \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \langle D_{E_0}^2 f(Y_t^x) [E_0(X_t^x - Y_t^x)], E_0(X_t^x - Y_t^x) \rangle\right) J_{ij}^2(t, x)\right], \\ \tilde{\Lambda}_{22}(t, x) &= \mathbb{E}\left[\left(\langle D_{E_0} f(Y_t^x), E_0(X_t^x - Y_t^x) \rangle \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \langle D_{E_0}^2 f(Y_t^x) [E_0(X_t^x - Y_t^x)], E_0(X_t^x - Y_t^x) \rangle\right) J_{ij}^2(t, x)\right], \end{aligned}$$

$t \in (0, 1)$, $x \in \mathbb{R}^n$. We have (using (3.7), (3.8), Lemmas 3.5 and 3.7)

$$\sup_{x \in \mathbb{R}^n} |\tilde{\Lambda}_1(t, x)| \leq c \|f\|_{\gamma, d} \sup_{x \in \mathbb{R}^n} \mathbb{E}\left[\left\{\sum_{m=1}^k |E_m(X_t^x - Y_t^x)|^{\frac{\gamma}{2m+1}}\right\} |J_{ij}^2(t, x)|\right] \leq c_2 \|f\|_{\gamma, d} t^{\frac{\gamma-2}{2} - h - h'}.$$

By the mean value theorem, we find

$$\sup_{x \in \mathbb{R}^n} |\tilde{\Lambda}_{21}(t, x)| \leq \|f\|_{\gamma, d} \sup_{x \in \mathbb{R}^n} \{\mathbb{E}[|E_0(X_t^x - Y_t^x)|^\gamma |J_{ij}^2(t, x)|]\} \leq c \|f\|_{\gamma, d} t^{\frac{2\gamma-2}{2} - h - h'}.$$

Using Lemma 3.7, we infer $\sup_{x \in \mathbb{R}^n, t \in (0,1)} |\tilde{\Lambda}_{22}(t, x)| = c_5 < \infty$ and this gives the assertion.

- We prove the estimate (iv) in (3.10), for $t \in (0, 1)$ and $\gamma \in (2, 3)$.

We have to show that, for any h , $0 \leq h \leq k$,

$$\sup_{x \in \mathbb{R}^n} \|P_t f(x + \cdot)\|_{C_b^{\gamma/2h+1}(E_h(\mathbb{R}^n))} \leq c \|f\|_{\gamma, d}, \quad f \in \mathcal{C}_d^\gamma(\mathbb{R}^n), \quad t \in (0, 1). \quad (3.26)$$

Fix the integer h , $f \in \mathcal{C}_d^\gamma(\mathbb{R}^n)$ and consider $\Delta_{v_h}^3(P_t f)(x) = P_t f(x) - 3P_t f(x + v_h) + 3P_t f(x + 2v_h) - P_t f(x + 3v_h)$, for $x \in \mathbb{R}^n$, $v_h \in E_h(\mathbb{R}^n)$ with $|v_h| \leq 1$ and $v_h \neq 0$. By (2.8) the assertion (3.26) is equivalent to the estimate

$$\sup_{x \in \mathbb{R}^n} |\Delta_{v_h}^3(P_t f)(x)| \leq c_1 \|f\|_{\gamma, d} |v_h|^{\frac{\gamma}{2h+1}}, \quad t \in (0, 1), \quad (3.27)$$

where c_1 is independent on f , t and v_h . We prove (3.27) considering first the case of $|v_h| \leq t^{\frac{2h+1}{2}} < 1$ and then the case of $1 \geq |v_h| > t^{\frac{2h+1}{2}}$ (compare with [16, page 148]).

(a) Let $|v_h| \leq t^{\frac{2h+1}{2}} < 1$. Using the mean value theorem and (iii) in (3.10), we get

$$\begin{aligned} &\sup_{x \in \mathbb{R}^n} |\Delta_{v_h}^3(P_t f)(x)| \leq \sup_{x \in \mathbb{R}^n, i, j, r \in I_h} \|D_{ijr}^3 P_t f\|_0 |v_h|^3 \\ &\leq c \frac{1}{t^{3h + \frac{3-\gamma}{2}}} \|f\|_{\gamma, d} |v_h|^3 \leq c \frac{1}{|v_h|^{\frac{2}{2h+1} \frac{6h+3-\gamma}{2}}} \|f\|_{\gamma, d} |v_h|^3 = c \|f\|_{\gamma, d} |v_h|^{\frac{\gamma}{2h+1}}, \quad t \in (0, 1). \end{aligned}$$

(b) Let $1 \geq |v_h| > t^{\frac{2h+1}{2}}$. We first estimate $\|e^{tA}v_h\|$. To this purpose we use that

$$\|E_i e^{tA} E_h\|_L \leq ct^{i-h}, \quad 0 \leq h \leq i \leq k; \quad \|E_i e^{tA} E_h\|_L \leq ct, \quad 0 \leq i < h \leq k, \quad t \in [0, 1] \quad (3.28)$$

(see [16, Lemma 3.1]) where $c = c(A) > 0$. Since $t \leq |v_h|^{\frac{2}{2h+1}} \leq 1$, we get

$$\begin{aligned} \|e^{tA}v_h\| &= \sum_{i=0}^{h-1} |E_i e^{tA} E_h v_h|^{\frac{1}{2i+1}} + \sum_{i=h}^k |E_i e^{tA} E_h v_h|^{\frac{1}{2i+1}} \leq c_1 \sum_{i=0}^{h-1} |tv_h|^{\frac{1}{2i+1}} \\ &+ c_1 \sum_{i=h}^k t^{\frac{i-h}{2i+1}} |v_h|^{\frac{1}{2i+1}} \leq c_1 h |v_h|^{1/(2h+1)} + c_1 \sum_{i=h}^k |v_h|^{\frac{1}{2i+1} + \frac{i-h}{2i+1} \frac{2}{2h+1}} \leq c_1 (k+1) |v_h|^{\frac{1}{2h+1}}. \end{aligned} \quad (3.29)$$

To finish the proof we will use the Girsanov theorem, see (3.4). First note that

$$\begin{aligned} &\Delta_{v_h}^3 (P_t f)(x) \\ &= \mathbb{E} \left[f(Z_t^x) \Phi(t, x) - 3f(Z_t^{x+v_h}) \Phi(t, x+v_h) \right. \\ &\quad \left. + 3f(Z_t^{x+2v_h}) \Phi(t, x+2v_h) - f(Z_t^{x+3v_h}) \Phi(t, x+3v_h) \right] \\ &= A_1(t, x) + A_2(t, x), \quad \text{where} \\ A_1(t, x) &= \mathbb{E} \left[\left(f(Z_t^x) - 3f(Z_t^{x+v_h}) + 3f(Z_t^{x+2v_h}) - f(Z_t^{x+3v_h}) \right) \Phi(t, x) \right], \\ A_2(t, x) &= 3\mathbb{E} \left[f(Z_t^{x+v_h}) (\Phi(t, x) - \Phi(t, x+v_h)) \right] + 3\mathbb{E} \left[f(Z_t^{x+2v_h}) (\Phi(t, x+2v_h) - \Phi(t, x)) \right] \\ &\quad + \mathbb{E} \left[f(Z_t^{x+3v_h}) (\Phi(t, x) - \Phi(t, x+3v_h)) \right]. \end{aligned}$$

Let us consider A_1 . We find, for any $x \in \mathbb{R}^n$, $t \in (0, 1)$, thanks to Lemma 2.1,

$$|A_1(t, x)| \leq \mathbb{E} \left[|\Delta_{e^{tA}v_h}^3 f(e^{tA}x + Z_t^0)| \Phi(t, x) \right] \leq \|f\|_{\gamma, d} \|e^{tA}v_h\|^\gamma \leq c \|f\|_{\gamma, d} |v_h|^{\frac{\gamma}{2h+1}}$$

(in the last inequality we have used (3.29)). It remains to treat A_2 . We have:

$$\begin{aligned} A_2(t, x) &= A_{21}(t, x) + A_{22}(t, x), \quad \text{where} \\ A_{21}(t, x) &= \mathbb{E} \left[f(Z_t^x) \left(\Phi(t, x) - 3\Phi(t, x+v_h) + 3\Phi(t, x+2v_h) - \Phi(t, x+3v_h) \right) \right], \\ A_{22}(t, x) &= 3\mathbb{E} \left[(f(Z_t^{x+v_h}) - f(Z_t^x)) (\Phi(t, x) - \Phi(t, x+v_h)) \right] \\ &\quad + 3\mathbb{E} \left[(f(Z_t^{x+2v_h}) - f(Z_t^x)) (\Phi(t, x+2v_h) - \Phi(t, x)) \right] \\ &\quad + \mathbb{E} \left[(f(Z_t^{x+3v_h}) - f(Z_t^x)) (\Phi(t, x) - \Phi(t, x+3v_h)) \right]. \end{aligned}$$

In order to treat A_{21} , remark that the map: $x \mapsto \Phi(t, x)$ is three times Fréchet differentiable from \mathbb{R}^n with values in $L^1(\Omega)$. We need to estimate the norm of the first, second and third Fréchet derivatives of $\Phi(t, \cdot)$ (these Fréchet derivatives will be indicated with $D_x \Phi(t, x)$, $D_{xx}^2 \Phi(t, x)$ and $D_{xxx}^3 \Phi(t, x)$ respectively).

For any $x, h \in \mathbb{R}^n$, we find (setting $G = Q^{-1/2}F$)

$$\begin{aligned} D_x \Phi(t, x)[h] &= \Phi(t, x) \int_0^t \langle DG(Z_s^x) e^{sA} h, dL_s^x \rangle \\ &= \Phi(t, x) \left(\int_0^t \langle DG(Z_s^x) e^{sA} h, dW_s \rangle - \int_0^t \langle DG(Z_s^x) e^{sA} h, G(Z_s^x) \rangle ds \right), \end{aligned}$$

since $L_s^x := W_s - \int_0^s G(Z_r^x) dr$, $s \in [0, t]$ (see (3.4)). By the Girsanov theorem, we have

$$\mathbb{E}|D_x \Phi(t, x)[h]| = \mathbb{E} \left| \int_0^t \langle DG(X_s^x) e^{sA} h, dW_s \rangle \right| \leq e^{\|A\|_L} |h| t^{1/2} \|DG\|_0 \leq e^{\|A\|_L} |h| \|DG\|_0, \quad (3.30)$$

for $t \in [0, 1]$, $h \in \mathbb{R}^n$. It follows that $\|D_x \Phi(t, x)\|_{L(\mathbb{R}^n, L^1(\Omega))} \leq e^{\|A\|} \|Q_0^{-1/2}\|_{L(\mathbb{R}^{\tilde{p}})} \|DF\|_0$, $t \in [0, 1]$. Similarly, we have for the second Fréchet derivative

$$\begin{aligned} D_{xx}^2 \Phi(t, x)[h][k] &= \Phi(t, x) \left(\int_0^t \langle DG(Z_s^x) e^{sA} h, dL_s^x \rangle \right) \left(\int_0^t \langle DG(Z_s^x) e^{sA} k, dL_s^x \rangle \right) \\ &+ \Phi(t, x) \left(\int_0^t \langle D^2 G(Z_s^x)[e^{sA} k][e^{sA} h], dL_s^x \rangle - \int_0^t \langle DG(Z_s^x)[e^{sA} h], DG(Z_s^x)[e^{sA} k] \rangle ds \right), \end{aligned}$$

$h, k \in \mathbb{R}^n$. It follows, by the Girsanov theorem,

$$\mathbb{E}|D_{xx}^2 \Phi(t, x)[h][k]| \leq c_2 |h||k| (\|DG\|_0^2 + \|D^2 G\|_0), \quad \text{for any } t \in [0, 1], h, k \in \mathbb{R}^n.$$

In a similar way we get

$$\mathbb{E}|D_{xxx}^3 \Phi(t, x)[h][k][u]| \leq c |h||k||u| (\|DG\|_0^3 + \|DG\|_0^2 + \|D^2 G\|_0^2 + \|D^3 G\|_0) \leq C_1 |h||k||u|,$$

for any $t \in [0, 1]$, $h, k, u \in \mathbb{R}^n$, where $C_1 = C_1(\|A\|_L, \nu_1, \tilde{p}, \|DF\|_0, \|D^2 F\|_0, \|D^3 F\|_0) > 0$. Using the last estimate, we find

$$\begin{aligned} |A_{21}(t, x)| &\leq \|f\|_0 \|\Phi(t, x) - 3\Phi(t, x + v_h) + 3\Phi(t, x + 2v_h) - \Phi(t, x + 3v_h)\|_{L^1(\Omega)} \\ &\leq \|f\|_0 \sup_{|u| \leq 1, |h| \leq 1, |k| \leq 1, x \in \mathbb{R}^n} \|D_{xxx}^3 \Phi(t, x)[h][k][u]\|_{L^1(\Omega)} |v_h|^3 \leq C_1 \|f\|_0 |v_h|^3, \end{aligned}$$

$x \in \mathbb{R}^n$, $t \in [0, 1]$. It remains to consider A_{22} . This is the sum of three terms which can be treated in the same way. Let us estimate the first term (without the factor 3). By (3.29), we find (recall that $\gamma \in (2, 3)$)

$$\begin{aligned} &\mathbb{E} \left| (f(Z_t^{x+v_h}) - f(Z_t^x)) (\Phi(t, x) - \Phi(t, x + v_h)) \right| \\ &\leq \|f\|_{\gamma, d} \|e^{tA} v_h\| \mathbb{E} |\Phi(t, x) - \Phi(t, x + v_h)| \leq c \|f\|_{\gamma, d} |v_h|^{\frac{1}{2h+1}} \|\Phi(t, x) - \Phi(t, x + v_h)\|_{L^1(\Omega)}. \end{aligned}$$

By (3.30), since $|v_h| > t^{\frac{2h+1}{2}}$,

$$\begin{aligned} \mathbb{E} \left| (f(Z_t^{x+v_h}) - f(Z_t^x)) (\Phi(t, x) - \Phi(t, x + v_h)) \right| &\leq e^{\|A\|_L} \|f\|_{\gamma, d} |v_h|^{\frac{1}{2h+1}} |v_h| t^{1/2} \|DG\|_0 \\ &\leq c' |v_h|^{\frac{2}{2h+1} + 1} \|f\|_{\gamma, d}. \end{aligned}$$

We obtain $\sup_{x \in \mathbb{R}^n} |A_{22}(t, x)| \leq c_3 |v_h|^{\frac{3+2h}{2h+1}} \|f\|_{\gamma, d}$, $t \in (0, 1)$. Using the estimates obtained for $A_1(t, x)$ and $A_2(t, x)$, assertion (3.27) follows. This completes the proof. \blacksquare

4 Elliptic and parabolic Schauder estimates

Here we prove elliptic and parabolic Schauder estimates for \mathcal{A} using the L^∞ -estimates of the previous section. Our method is different with respect to [16, 4, 15, 27] (see Theorems 4.2 and 4.3). Before proving Schauder estimates, we show existence and uniqueness of distributional solutions for (1.2) and (1.3).

Let $\lambda > 0$ and $f \in C_b(\mathbb{R}^n)$ (i.e., f is uniformly continuous and bounded on \mathbb{R}^n). We say that a function $u \in C_b(\mathbb{R}^n)$ is a *distributional solution* to the elliptic equation

$$\lambda u(x) - \mathcal{A}u(x) = f(x), \quad x \in \mathbb{R}^n, \quad (4.1)$$

if $\lambda \int_{\mathbb{R}^n} u(x)\phi(x)dx = \int_{\mathbb{R}^n} u(x)\mathcal{A}^*\phi(x)dx + \int_{\mathbb{R}^n} f(x)\phi(x)dx$, for any $\phi \in C_0^\infty(\mathbb{R}^n)$, where \mathcal{A}^* is the formal adjoint of \mathcal{A} , i.e.,

$$\mathcal{A}^*\phi(x) = \frac{1}{2}\text{Tr}(QD^2\phi(x)) - \langle Ax + F(x), D\phi(x) \rangle - \phi(x)[\text{div}F(x) + \text{Tr}(A)], \quad x \in \mathbb{R}^n.$$

Let $g \in C_b(\mathbb{R}^n)$, $T > 0$ and $H : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous and bounded function. We say that a continuous and bounded function $v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $v(0, x) = g(x)$, $x \in \mathbb{R}^n$, is a *space-distributional solution* to the parabolic Cauchy problem

$$\begin{cases} \partial_t v(t, x) = \mathcal{A}v(t, x) + H(t, x), & t \in (0, T], x \in \mathbb{R}^n, \\ v(0, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (4.2)$$

if the following conditions hold:

- (i) $v(t, \cdot) \in C_b(\mathbb{R}^n)$ uniformly in $t \in [0, T]$ (i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in \mathbb{R}^n$ and $|y| < \delta$, we have $\sup_{t \in [0, T], x \in \mathbb{R}^n} |v(t, x + y) - v(t, x)| < \epsilon$);
- (ii) for any test function $\phi \in C_0^\infty(\mathbb{R}^n)$, the real mapping: $t \mapsto \int_{\mathbb{R}^n} v(t, x)\phi(x)dx$ is continuously differentiable on $[0, T]$ and moreover

$$\frac{d}{dt} \left(\int_{\mathbb{R}^n} v(t, x)\phi(x)dx \right) = \int_{\mathbb{R}^n} v(t, x)\mathcal{A}^*\phi(x)dx + \int_{\mathbb{R}^n} H(t, x)\phi(x)dx, \quad t \in [0, T]. \quad (4.3)$$

Theorem 4.1. *Let $\lambda > 0$ and $f \in C_b(\mathbb{R}^n)$. Then there exists a unique distributional solution $u \in C_b(\mathbb{R}^n)$ to the equation (4.1). Moreover u is given by*

$$u(x) = \int_0^\infty e^{-\lambda t} (P_t f)(x) dt = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad x \in \mathbb{R}^n, \quad (4.4)$$

where P_t is the diffusion semigroup introduced in (3.2).

Let $g \in C_b(\mathbb{R}^n)$, $T > 0$ and $H : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous and bounded. Then there exists a unique space-distributional solution v to the Cauchy problem (4.2). Moreover, setting $\int_0^t P_{t-s} H(s, x) ds := \int_0^t P_{t-s}(H(s, \cdot))(x) ds$, we have

$$v(t, x) = P_t g(x) + \int_0^t P_{t-s} H(s, x) ds, \quad x \in \mathbb{R}^n, t \in [0, T]. \quad (4.5)$$

Proof. Uniqueness. We first consider the *elliptic case*. Fix $\lambda > 0$ and let $u \in C_b(\mathbb{R}^n)$ be any distributional solution to (4.1) with $f = 0$.

Take a function $\rho \in C_0^\infty(\mathbb{R}^n)$ such that $\|\rho\|_{L^1(\mathbb{R}^n)} = 1$, $0 \leq \rho \leq 1$ and $\rho(x) = 0$ if $|x| \geq 1$. Define a sequence of mollifiers $(\rho_m) \subset C_0^\infty(\mathbb{R}^n)$, $\rho_m(x) := m^n \rho(mx)$, $x \in \mathbb{R}^n$, $m \in \mathbb{N}$. Consider the functions $u_m \in C_b^\infty(\mathbb{R}^n)$ obtained by convolution of u with ρ_m , i.e., $u_m = u * \rho_m$. Setting $B(x) := Ax + F(x)$, $x \in \mathbb{R}^n$, we use the identity:

$$\mathcal{A}^*[\rho_m(x - \cdot)](y) + \langle B(x) - B(y), D\rho_m(x - y) \rangle + \rho_m(x - y)\text{div}B(y) = \mathcal{A}[\rho_m(\cdot - y)](x),$$

$x, y \in \mathbb{R}^n$, and get

$$\begin{aligned}
\mathcal{A}u_m(x) &= \int_{\mathbb{R}^n} u(y) \mathcal{A}[\rho_m(\cdot - y)](x) dy = \int_{\mathbb{R}^n} u(y) \mathcal{A}^*[\rho_m(x - \cdot)](y) dy + R_{m,1}(x) + R_{m,2}(x) \\
&= \int_{\mathbb{R}^n} \lambda u(y) \rho_m(x - y) dy + R_{m,1}(x) + R_{m,2}(x), \quad \text{where} \\
R_{m,1}(x) &= \int_{\mathbb{R}^n} u(y) \operatorname{div} B(y) \rho_m(x - y) dy, \\
R_{m,2}(x) &= \int_{\mathbb{R}^n} u(y) \langle B(x) - B(y), D\rho_m(x - y) \rangle dy.
\end{aligned} \tag{4.6}$$

Changing variable as in [15, page 559], we obtain

$$R_{m,2}(x) = m \int_{\mathbb{R}^n} u(x - \frac{z}{m}) \langle B(x) - B(x - \frac{z}{m}), D\rho(z) \rangle dz.$$

It follows that $R_{m,2}$ converges as $m \rightarrow \infty$, *uniformly on* \mathbb{R}^n , to the function

$$x \mapsto u(x) \sum_{i,k=1}^n \int_{\mathbb{R}^n} D_k B_i(x) z_k D_i \rho(z) dz = -u(x) \operatorname{div} B(x).$$

On the other hand, it is easy to see that $R_{m,1}$ converges as $m \rightarrow \infty$, uniformly on \mathbb{R}^n , to $u \operatorname{div} B$. It follows that $\lim_{m \rightarrow \infty} (R_{m,1} + R_{m,2}) = 0$ in $C_b(\mathbb{R}^n)$. Hence we have obtained

$$\lim_{m \rightarrow \infty} (\|\mathcal{A}u_m - \lambda u\|_0 + \|u_m - u\|_0) = 0.$$

By the classical maximum principle (see [19]) we deduce that $\|u_m\|_0 \leq \frac{1}{\lambda} \|\lambda u_m - \mathcal{A}u_m\|_0$. Letting $m \rightarrow \infty$, we find that $\|u\|_0 = 0$ and this gives the assertion.

We prove now uniqueness in the *parabolic case*. To this purpose, we take $H = 0$ and $g = 0$ in (4.2) and consider any space-distributional solution v . We introduce as before a sequence of mollifiers $(\rho_m) \subset C_0^\infty(\mathbb{R}^n)$ and define

$$v_m(t, x) = \int_{\mathbb{R}^n} v(t, y) \rho_m(x - y) dy, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad m \in \mathbb{N}.$$

It is clear that v_m is continuous and bounded on $[0, T] \times \mathbb{R}^n$. Moreover, there exist continuous and bounded spatial partial derivatives of v_m of any order on $[0, T] \times \mathbb{R}^n$. Thanks to assumption (i), v_m converges to v as $m \rightarrow \infty$ *uniformly* on $[0, T] \times \mathbb{R}^n$.

We have, by (4.3), for $t \in [0, T]$, $x \in \mathbb{R}^n$,

$$\begin{aligned}
\partial_t v_m(t, x) &= \int_{\mathbb{R}^n} v(t, y) \mathcal{A}^*[\rho_m(x - \cdot)](y) dy = \int_{\mathbb{R}^n} v(t, y) \mathcal{A}[\rho_m(\cdot - y)](x) dy \\
&+ S_{m,1}(t, x) + S_{m,2}(t, x) = \mathcal{A}v_m(t, x) + S_{m,1}(t, x) + S_{m,2}(t, x), \quad \text{where} \\
S_{m,1}(t, x) &= - \int_{\mathbb{R}^n} v(t, y) \operatorname{div} B(y) \rho_m(x - y) dy, \\
S_{m,2}(t, x) &= - \int_{\mathbb{R}^n} v(t, y) \langle B(x) - B(y), D\rho_m(x - y) \rangle dy.
\end{aligned} \tag{4.7}$$

Remark that $\lim_{m \rightarrow \infty} \sup_{t \in [0, T], x \in \mathbb{R}^n} |S_{m,1}(t, x) + S_{m,2}(t, x)| = 0$. Moreover, since v_m is a classical solution to

$$\begin{cases} \partial_t v_m(t, x) = \mathcal{A}v_m(t, x) + S_{m,1}(t, x) + S_{m,2}(t, x), & t \in (0, T], x \in \mathbb{R}^n, \\ v_m(0, x) = 0, & x \in \mathbb{R}^n. \end{cases}$$

by the classical parabolic maximum principle (see [10, Chapter 8]) we have

$$\sup_{t \in [0, T], x \in \mathbb{R}^n} |v_m(t, x)| \leq T \sup_{t \in [0, T], x \in \mathbb{R}^n} |S_{m,1}(t, x) + S_{m,2}(t, x)|.$$

Letting $m \rightarrow \infty$ we obtain that $v = 0$ and this proves the assertion.

Existence. We first consider the *elliptic case* and prove that u given in (4.4) is the distributional solution. It is clear that $u \in C_b(\mathbb{R}^n)$. In the following computations we will use that there exists the classical partial derivative $\partial_t(P_t f)(x)$, for $t > 0$ and $x \in \mathbb{R}^n$, and $\partial_t(P_t f)(x) = \mathcal{A}(P_t f)(x)$, see [25, Section 4].

By Corollary 3.2 we deduce that, for any $M > 0$, there exists $C_M > 0$ such that

$$\sup_{|x| \leq M} |\mathcal{A}(P_t f)(x)| \leq C_M (t^{-(1+k)} + 1) \|f\|_0, \quad t > 0, f \in C_b(\mathbb{R}^n). \quad (4.8)$$

We obtain, for any $\phi \in C_0^\infty(\mathbb{R}^n)$, applying the Fubini theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} u(x) \mathcal{A}^* \phi(x) &= \int_0^\infty e^{-\lambda t} dt \int_{\mathbb{R}^n} \mathcal{A}P_t f(x) \phi(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_\epsilon^\infty e^{-\lambda t} dt \int_{\mathbb{R}^n} \mathcal{A}P_t f(x) \phi(x) dx \\ &= \lim_{\epsilon \rightarrow 0^+} \int_\epsilon^\infty e^{-\lambda t} dt \int_{\mathbb{R}^n} \partial_t P_t f(x) \phi(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} \phi(x) dx \int_\epsilon^\infty e^{-\lambda t} \partial_t P_t f(x) dt \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n} \left(-e^{-\lambda \epsilon} P_\epsilon f(x) + \lambda \int_\epsilon^\infty e^{-\lambda t} P_t f(x) dt \right) \phi(x) = \int_{\mathbb{R}^n} (-f(x) + \lambda u(x)) \phi(x) dx. \end{aligned}$$

We deal now with the *parabolic case* and show that v given in (4.5) is the space-distributional solution. We write

$$v = v_1 + v_2, \quad \text{where } v_1(t, x) = P_t g(x), \quad v_2(t, x) = \int_0^t P_{t-s} H(s, x) ds, \quad (4.9)$$

$v_2(0, \cdot) = 0$ (v_1 and v_2 are associated to (4.5) when $H = 0$ and $g = 0$ respectively). First we deal with v_1 . In [25, Section 4] it is verified that v_1 is a continuous and bounded function on $[0, \infty) \times \mathbb{R}^n$. Moreover, denoting by ω_g the modulus of continuity of g , we have, for any $t \in [0, T]$, $x, y \in \mathbb{R}^n$, $|P_t g(x) - P_t g(y)| \leq \mathbb{E}[\omega_g(|X_t^x - X_t^y|)] \leq \omega_g(|x - y| e^{TL})$, where $L = \|A\|_L + \|DF\|_0$. This shows that $v_1(t, \cdot) \in C_b(\mathbb{R}^n)$, uniformly in $t \in [0, T]$.

Since it holds (in a classical sense) $\partial_t(P_t f)(x) = \mathcal{A}(P_t f)(x)$, $t > 0$, $x \in \mathbb{R}^n$, we have that $t \mapsto \int_{\mathbb{R}^n} v_1(t, x) \phi(x) dx$ belongs to $C^1([0, T])$ and verifies (4.3) (with $H = 0$).

Let us treat v_2 . By the first estimate in (3.9) we deduce, for any $f : \mathbb{R}^n \rightarrow \mathbb{R}$ continuous and bounded, for any $h \in \{0, \dots, k\}$,

$$\|P_t f(x + \cdot)\|_{C_b^{\frac{1}{2k+1}}(E_h(\mathbb{R}^n))} \leq \|f\|_0^{1 - \frac{1}{2k+1}} \|P_t f(x + \cdot)\|_{C_b^1(E_h(\mathbb{R}^n))}^{\frac{1}{2k+1}} \leq C_1 t^{-1/2} \|f\|_0, \quad t \in (0, T],$$

$x \in \mathbb{R}^n$, where C_1 is independent on t, x and f . It follows that, for any $x, y \in \mathbb{R}^n$, $t \in [0, T]$,

$$|v_2(t, x) - v_2(t, y)| \leq \int_0^t \frac{C_1 \|H\|_0}{(t-s)^{1/2}} ds \sum_{h=0}^k |E_h(x-y)|^{\frac{1}{2k+1}} \leq c' \|H\|_0 \sqrt{T} |x-y|^{\frac{1}{2k+1}},$$

where $\|H\|_0 = \sup_{t \in [0, T], x \in \mathbb{R}^n} |H(t, x)|$. This shows that $v_2(t, \cdot) \in C_b(\mathbb{R}^n)$, uniformly in t . By this property, in order to verify that v_2 is continuous on $[0, T] \times \mathbb{R}^n$, it is enough to check that for any fixed $x \in \mathbb{R}^n$, $v_2(\cdot, x)$ is continuous on $[0, T]$. Since the continuity of $v_2(\cdot, x)$ in $t = 0$ is clear, we consider continuity at a fixed $t \in (0, T]$. We write, for h sufficiently small,

$$v_2(t+h, x) - v_2(t, x) = \int_0^T [P_{t+h-s} H(s, x) - P_{t-s} H(s, x)] ds. \quad (4.10)$$

(we have extended P_t to negative values, setting $P_\eta = 0$, $\eta < 0$). By the dominated convergence theorem one deduces that $\lim_{h \rightarrow 0} v_2(t+h, x) = v_2(t, x)$. Thus v_2 is continuous on $[0, T] \times \mathbb{R}^n$ and $v_2(0, \cdot) = 0$. The boundedness of v_2 is clear.

It remains to verify that v_2 satisfy (4.3). To this purpose, we fix $t \in (0, T]$, $x \in \mathbb{R}^n$, and consider for $h > 0$ (see also [24, pages 58-59])

$$\begin{aligned} \frac{v_2(t+h, x) - v_2(t, x)}{h} &= \Gamma_1(t, h, x) + \Gamma_2(t, h, x), \\ \Gamma_1(t, h, x) &= \frac{1}{h} \int_t^{t+h} P_{t+h-s} H(s, x) ds, \quad \Gamma_2(t, h, x) = \int_0^t \left(\frac{P_{t+h-s} - P_{t-s}}{h} \right) H(s, x) ds. \end{aligned}$$

We have: $|\Gamma_1(t, h, x) - H(t, x)| \leq \int_0^1 \mathbb{E} |H(t+h-sh, X_{sh}^x) - H(t, x)| ds \rightarrow 0$ as h tends to 0^+ , by the dominated convergence theorem. It follows that, for any $\phi \in C_0^\infty(\mathbb{R}^n)$, $\lim_{h \rightarrow 0^+} \int_{\mathbb{R}^n} \Gamma_1(t, h, x) \phi(x) dx = \int_{\mathbb{R}^n} H(t, x) \phi(x) dx$.

Concerning Γ_2 , we first note that, thanks to (4.8), for any $t > s \geq 0$,

$$\lim_{h \rightarrow 0^+} \int_{\mathbb{R}^n} \left(\frac{P_{t+h-s} H(s, x) - P_{t-s} H(s, x)}{h} \right) \phi(x) dx = \int_{\mathbb{R}^n} \mathcal{A}[P_{t-s} H(s, \cdot)](x) \phi(x) dx.$$

$$\begin{aligned} \text{By the Fubini theorem we get } \lim_{h \rightarrow 0^+} \int_{\mathbb{R}^n} \Gamma_2(t, h, x) \phi(x) dx &= \\ = \int_0^t ds \int_{\mathbb{R}^n} P_{t-s} H(s, x) \mathcal{A}^* \phi(x) dx &= \int_{\mathbb{R}^n} \mathcal{A}^* \phi(x) dx \int_0^t P_{t-s} H(s, x) ds, \quad t \in]0, T]. \end{aligned}$$

It follows easily that the map $t \mapsto \int_{\mathbb{R}^n} v_2(t, x) \phi(x)$ belongs to $C^1([0, T])$ and verifies (4.3) (with $g = 0$) for $t \in [0, T]$. This finishes the proof. \blacksquare

The next theorems provide elliptic and parabolic Schauder estimates.

Theorem 4.2. *Let $\theta \in (0, 1)$ and $\lambda > 0$. For any $f \in C_d^\theta(\mathbb{R}^n)$ there exists a unique distributional solution to the elliptic equation (4.1). Moreover $u \in C_d^{2+\theta}(\mathbb{R}^n)$ and there exists $c = c(\lambda, \theta, \nu_1, \nu_2, A, \tilde{p}, n, \|DF\|_0, \|D^2F\|_0, \|D^3F\|_0)$, such that*

$$\|u\|_{2+\theta, d} \leq c \|f\|_{\theta, d}. \quad (4.11)$$

Proof. Uniqueness follows by Theorem 4.1. To prove the assertion we need to investigate the regularity properties of the function $u \in C_b(\mathbb{R}^n)$ given in (4.4).

We first prove that $u(z + \cdot) \in C_b^{2+\theta}(E_0(\mathbb{R}^n))$, for any $z \in \mathbb{R}^n$, and

$$\sup_{z \in \mathbb{R}^n} \|u(z + \cdot)\|_{C_b^{2+\theta}(E_0(\mathbb{R}^n))} \leq C \|f\|_{\theta, d}. \quad (4.12)$$

It is clear by the estimates (3.11) that there exist the partial derivatives $D_i u$ and $D_{ij}^2 u$ on \mathbb{R}^n , for any $i, j \in I_0 = \{1, \dots, \tilde{p}\}$. Moreover $D_i u$ and $D_{ij}^2 u$ are continuous and bounded on \mathbb{R}^n and $\|D_i u\|_0 + \|D_{ij}^2 u\|_0 \leq c \|f\|_{\theta, d}$.

We will prove now that $D_{ij}^2 u \in C_d^\theta(\mathbb{R}^n)$ when $i, j \in I_0$. This will imply (4.12). To this purpose, we fix any $v_h \in E_h(\mathbb{R}^n)$, for $0 \leq h \leq k$, with $|v_h| \leq 1$, and compute

$$\begin{aligned} |D_{ij}^2 u(x + v_h) - D_{ij}^2 u(x)| &\leq \int_0^\infty e^{-\lambda t} |D_{ij}^2 P_t f(x + v_h) - D_{ij}^2 P_t f(x)| dt = u_1(x) + u_2(x), \\ u_1(x) &= \int_0^{|v_h|^{\frac{2}{2h+1}}} e^{-\lambda t} |D_{ij}^2 P_t f(x + v_h) - D_{ij}^2 P_t f(x)| dt; \\ u_2(x) &= \int_{|v_h|^{\frac{2}{2h+1}}}^\infty e^{-\lambda t} |D_{ij}^2 P_t f(x + v_h) - D_{ij}^2 P_t f(x)| dt, \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.13}$$

In order to estimate $u_1(x)$ we use (b) in (3.11). We find

$$\sup_{x \in \mathbb{R}^n} u_1(x) \leq c \|f\|_{\theta, d} \int_0^{|v_h|^{\frac{2}{2h+1}}} t^{\frac{\theta}{2}-1} dt \leq C \|f\|_{\theta, d} |v_h|^{\frac{\theta}{2h+1}}.$$

Concerning $u_2(x)$ we use estimate (c) in (3.11). This gives, for any $x \in \mathbb{R}^n$,

$$|D_{ij}^2 P_t f(x + v_h) - D_{ij}^2 P_t f(x)| \leq |v_h| \sup_{r \in I_h} \|D_{ijr}^3 P_t f\|_0 \leq c \|f\|_{\theta, d} \left(\frac{1}{t^{\frac{3-\theta}{2}+h}} + 1 \right) |v_h|, \quad t > 0.$$

We get

$$\begin{aligned} \sup_{x \in \mathbb{R}^n} u_2(x) &\leq c \|f\|_{\theta, d} |v_h| \int_{|v_h|^{\frac{2}{2h+1}}}^\infty e^{-\lambda t} (t^{\frac{\theta}{2}-\frac{3}{2}-h} + 1) dt \leq c' \left(\frac{|v_h|}{\lambda} + |v_h|^{\frac{\theta}{2h+1}} \right) \|f\|_{\theta, d} \\ &\leq C_1 \|f\|_{\theta, d} |v_h|^{\frac{\theta}{2h+1}}. \end{aligned}$$

It follows that $\sup_{x \in \mathbb{R}^n} |D_{ij}^2 u(x + v_h) - D_{ij}^2 u(x)| \leq C \|f\|_{\theta, d} |v_h|^{\frac{\theta}{2h+1}}$ and so (4.12) is proved.

We verify that $u(z + \cdot) \in C_b^{\frac{2+\theta}{2h+1}}(E_h(\mathbb{R}^n))$, for any $1 \leq h \leq k$, and moreover

$$\sup_{z \in \mathbb{R}^n} \|u(z + \cdot)\|_{C_b^{\frac{2+\theta}{2h+1}}(E_h(\mathbb{R}^n))} \leq C \|f\|_{\theta, d}. \tag{4.14}$$

We fix $v_h \in E_h(\mathbb{R}^n)$, for $1 \leq h \leq k$, with $|v_h| \leq 1$, and compute

$$\begin{aligned} |u(x + v_h) - u(x)| &\leq \int_0^\infty e^{-\lambda t} |P_t f(x + v_h) - P_t f(x)| dt = u_1(x) + u_2(x), \quad \text{where} \\ u_1(x) &= \int_0^{|v_h|^{\frac{2}{2h+1}}} e^{-\lambda t} |P_t f(x + v_h) - P_t f(x)| dt; \tag{4.15} \\ u_2(x) &= \int_{|v_h|^{\frac{2}{2h+1}}}^\infty e^{-\lambda t} |P_t f(x + v_h) - P_t f(x)| dt, \quad x \in \mathbb{R}^n. \end{aligned}$$

In order to estimate $u_1(x)$ we use (d) in (3.11). We find

$$\sup_{x \in \mathbb{R}^n} u_1(x) \leq c \|f\|_{\theta, d} |v_h|^{\frac{\theta}{2h+1}} \int_0^{|v_h|^{\frac{2}{2h+1}}} dt \leq C \|f\|_{\theta, d} |v_h|^{\frac{2+\theta}{2h+1}}.$$

Concerning $u_2(x)$ we use estimate (a) in (3.11). We get (recall that $h \geq 1$)

$$\sup_{x \in \mathbb{R}^n} u_2(x) \leq c \|f\|_{\theta, d} |v_h| \int_{|v_h|^{\frac{2}{2h+1}}}^\infty e^{-\lambda t} (t^{\frac{\theta}{2}-\frac{1}{2}-h} + 1) dt \leq C_1 \|f\|_{\theta, d} |v_h|^{\frac{2+\theta}{2h+1}}$$

and (4.14) follows. The proof is complete. ■

Theorem 4.3. Let $\theta \in (0, 1)$, $T > 0$, $g \in C_d^{2+\theta}(\mathbb{R}^n)$ and let $H : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function such that $\sup_{t \in [0, T]} \|H(t, \cdot)\|_{\theta, d} < \infty$.

Then the Cauchy problem (4.2) has a unique space-distributional solution v such that $v(t, \cdot) \in C_d^{2+\theta}(\mathbb{R}^n)$, $t \in [0, T]$. Moreover, $D_i v$ and $D_{ij}^2 v$ are continuous on $[0, T] \times \mathbb{R}^n$, for $i, j \in I_0$, and there exists $c = c(T, \theta, \nu_1, \nu_2, A, \tilde{p}, n, \|DF\|_0, \|D^2 F\|_0, \|D^3 F\|_0)$, such that

$$\sup_{t \in [0, T]} \|v(t, \cdot)\|_{2+\theta, d} \leq c(\|g\|_{2+\theta, d} + \sup_{t \in [0, T]} \|H(t, \cdot)\|_{\theta, d}). \quad (4.16)$$

Proof. Uniqueness follows by Theorem 4.1. To prove the result, we need to investigate the space-regularity of the function v given in (4.5); we write $v = v_1 + v_2$ as in (4.9).

Concerning the function $v_1 = P_t g$ the estimate (iv) in (3.10) with $\gamma = 2 + \theta$ gives immediately (4.16) with v replaced by v_1 and $H = 0$. In order to treat v_2 ,

$$v_2(t, x) = \int_0^t \mathbb{E}[H(s, X_{t-s}^x)] ds = \int_0^t \mathbb{E}[H(t-s, X_s^x)] ds, \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

we proceed as in the proof of Theorem 4.2. To this purpose, set $\|H\|_{T, \theta} = \sup_{t \in [0, T]} \|H(t, \cdot)\|_{\theta, d}$. We first prove that $v_2(t, z + \cdot) \in C_b^{2+\theta}(E_0(\mathbb{R}^n))$, for $t \in [0, T]$ and $z \in \mathbb{R}^n$, and that

$$\sup_{t \in [0, T], z \in \mathbb{R}^n} \|v_2(t, z + \cdot)\|_{C_b^{2+\theta}(E_0(\mathbb{R}^n))} \leq C \|H\|_{T, \theta}. \quad (4.17)$$

It is clear by the estimates (3.11) that there exist the spatial partial derivatives $D_i v_2$ and $D_{ij}^2 v_2$ on $[0, T] \times \mathbb{R}^n$, for any $i, j \in I_0$. Moreover $D_i v_2(t, \cdot)$ and $D_{ij}^2 v_2(t, \cdot)$ are continuous and bounded on \mathbb{R}^n and $\|D_i v_2(t, \cdot)\|_0 + \|D_{ij}^2 v_2(t, \cdot)\|_0 \leq c \|H\|_{T, \theta}$, for any $t \in [0, T]$.

To prove assertion (4.17), we fix $v_h \in E_h(\mathbb{R}^n)$, for $0 \leq h \leq k$, with $|v_h| \leq 1$, and compute as in (4.13), for any $x \in \mathbb{R}^n$,

$$\begin{aligned} & |D_{ij}^2 v_2(t, x + v_h) - D_{ij}^2 v_2(t, x)| \leq \int_0^t |D_{ij}^2 P_s H(t-s, x + v_h) - D_{ij}^2 P_s H(t-s, x)| ds \\ & \leq c \|H\|_{T, \theta} \int_0^{t \wedge |v_h|^{\frac{2}{2h+1}}} s^{\frac{\theta}{2}-1} ds + c_1 \|H\|_{T, \theta} |v_h| \int_{t \wedge |v_h|^{\frac{2}{2h+1}}}^t s^{\frac{\theta}{2}-\frac{3}{2}-h} ds \leq c' \|H\|_{T, \theta} |v_h|^{\frac{\theta}{2h+1}} \end{aligned}$$

($a \wedge b = \min(a, b)$) and so the assertion (4.17) is proved. In order to verify that $v_2(t, z + \cdot) \in C_b^{\frac{2+\theta}{2h+1}}(E_h(\mathbb{R}^n))$, for any $1 \leq h \leq k$, $t \in [0, T]$, and $\sup_{z \in \mathbb{R}^n, t \in [0, T]} \|v_2(t, z + \cdot)\|_{C_b^{\frac{2+\theta}{2h+1}}(E_h(\mathbb{R}^n))} \leq C \|H\|_{T, \theta}$, we proceed as in (4.15).

To prove the continuity of $D_i v$ and $D_{ij}^2 v$ on $[0, T] \times \mathbb{R}^n$, $i, j \in I_0$, it is enough to show that, for any fixed $x \in \mathbb{R}^n$, $D_i v(\cdot, x)$ and $D_{ij}^2 v(\cdot, x)$ are continuous on $[0, T]$. To this purpose, we write $x = x_0 + x_1$, where $x_0 = E_0 x$ and $x_1 = x - E_0 x$, and consider the closed euclidean ball K in $E_0(\mathbb{R}^n)$ centered in x_0 with radius 1. We already know that $\|v(t, x_1 + \cdot)\|_{C^{2+\theta}(K)} \leq C_T$, for any $t \in [0, T]$. Using the continuity of v on $[0, T] \times \mathbb{R}^n$ and a standard compactness argument we obtain the assertion. Note that in particular $\lim_{t \rightarrow 0^+} D_i v(t, x) = D_i g(x)$ and $\lim_{t \rightarrow 0^+} D_{ij}^2 v(t, x) = D_{ij}^2 g(x)$, $x \in \mathbb{R}^n$. \blacksquare

5 Schauder estimates with variable coefficients (q_{ij})

Here we study a generalization of the operator \mathcal{A} . Namely, we deal with the operator $\tilde{\mathcal{A}}$ in which the diffusion matrix Q depends continuously on x , i.e.,

$$\tilde{\mathcal{A}}u(x) = \frac{1}{2}\text{Tr} (Q(x)D^2u(x)) + \langle Ax, Du(x) \rangle + \langle F(x), Du(x) \rangle, \quad x \in \mathbb{R}^n. \quad (5.1)$$

Thanks to the results of Section 4, we will obtain elliptic and parabolic Schauder estimates for the operator $\tilde{\mathcal{A}}$, using a standard approach. This is based on maximum principle, a priori estimates and continuity method (compare with [16, Section 6]).

Hypothesis 5.1. (i) There exists $\nu > 0$ and an integer \tilde{p} , $1 \leq \tilde{p} \leq n$, such that the symmetric matrix $Q(x) = (q_{ij}(x))_{i,j=1,\dots,n}$ has the form

$$Q(x) = \begin{pmatrix} Q_0(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in \mathbb{R}^n, \quad (5.2)$$

where $Q_0(x)$ is a uniformly positive definite $\tilde{p} \times \tilde{p}$ -matrix such that

$$\nu \sum_{i=1}^{\tilde{p}} \xi_i^2 \leq \sum_{i,j=1}^{\tilde{p}} q_{ij}(x) \xi_i \xi_j \leq \frac{1}{\nu} \sum_{i=1}^{\tilde{p}} \xi_i^2, \quad \xi = (\xi_i) \in \mathbb{R}^{\tilde{p}}, \quad x \in \mathbb{R}^n. \quad (5.3)$$

(ii) The vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (ii) and (iii) in Hypothesis 1.1.

(iii) Assumption (iv) in Hypothesis 1.1 holds.

(iv) There exists $\theta \in (0, 1)$ such that $q_{ij} \in C_d^\theta(\mathbb{R}^n)$, for $i, j \in \{1, \dots, \tilde{p}\}$, and moreover there exists the limit

$$\lim_{|x| \rightarrow \infty} Q_0(x) = Q_0^\infty \text{ in } L(\mathbb{R}^{\tilde{p}}). \quad (5.4)$$

Let us comment on these assumptions. Note that, for every $x_0 \in \mathbb{R}^n$, the operator with frozen second order coefficients

$$\mathcal{A}(x_0) = \frac{1}{2}\text{Tr} (Q(x_0)D^2\cdot) + \langle F(x) + Ax, D\cdot \rangle \quad (5.5)$$

verifies Hypothesis 1.1 and therefore Theorems 4.2 and 4.3 hold for $\mathcal{A}(x_0)$. The same happens for the operator \mathcal{A}^∞ defined as in (5.5) but with $Q(x_0)$ replaced by Q^∞ (Q^∞ is the $n \times n$ matrix having Q_0^∞ in the first $\tilde{p} \times \tilde{p}$ block, and zero entries in the other blocks; clearly its coefficients q_{ij}^∞ verify (5.3)).

To prove the next theorems it is crucial to remark that the constants in the elliptic and parabolic Schauder estimates involving $\mathcal{A}(x_0)$ does not depend on $x_0 \in \mathbb{R}^n$.

Theorem 5.2. *Consider the operator $\tilde{\mathcal{A}}$ in (5.1) under Hypothesis 5.1. Then, for every $\lambda > 0$ and $f \in C_d^\theta(\mathbb{R}^n)$, the elliptic problem*

$$\lambda u - \tilde{\mathcal{A}}u = f \quad (5.6)$$

has a unique solution $u \in C_d^{2+\theta}(\mathbb{R}^n)$ (the first order term $\langle Ax, Du(x) \rangle$ in (5.1) is understood in distributional sense). Moreover there is $c > 0$, independent of f and u , such that Schauder estimates (4.11) hold for (5.6).

Proof. We will only sketch the proof which is not difficult. One needs first a *maximum principle* for (5.6). We explain how this result can be obtained arguing as in the proof of Theorem 4.1. We write $\tilde{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_1 = \frac{1}{2} \text{Tr}(Q(x)D^2 \cdot) \quad \text{and} \quad \mathcal{A}_2 = \langle F(x) + Ax, D \cdot \rangle. \quad (5.7)$$

Take any $u \in \mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$ which solves (5.6). Consider a sequence of mollifiers (ρ_m) and set $u_m = u * \rho_m$; we get, similarly to (4.6),

$$\tilde{\mathcal{A}}u_m(x) = \int_{\mathbb{R}^n} \mathcal{A}_1 u(x-y) \rho_m(y) dy + \int_{\mathbb{R}^n} u(y) \mathcal{A}_2^*[\rho_m(x-\cdot)](y) dy + R_{m,1}(x) + R_{m,2}(x),$$

$x \in \mathbb{R}^n$, $m \in \mathbb{N}$, where \mathcal{A}_2^* is the formal adjoint of \mathcal{A}_2 . One finds that $\tilde{\mathcal{A}}u_m$ converges in $C_b(\mathbb{R}^n)$ to $\tilde{\mathcal{A}}u$ as $m \rightarrow \infty$. By the classical maximum principle (see [19]) we deduce that $\|u_m\|_0 \leq \frac{1}{\lambda} \|\lambda u_m - \tilde{\mathcal{A}}u_m\|_0$. Letting $m \rightarrow \infty$, we find $\|u\|_0 \leq \frac{1}{\lambda} \|\lambda u - \tilde{\mathcal{A}}u\|_0$.

A priori estimates for (5.6) can be proved exactly as in the proof of [16, Theorem 8.1]. One assumes that $u \in \mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$ is a solution to (5.6) and then by using a localization argument and the maximum principle one finds that there exists $C = C > 0$ (independent on f and u) such that

$$\|u\|_{2+\theta,d} \leq C \|f\|_{\theta,d}.$$

The *continuity method* allows to conclude the proof. For any $\epsilon \in [0, 1]$ one considers the problem

$$\lambda u - (1 - \epsilon)\mathcal{A}^\infty u - \epsilon \tilde{\mathcal{A}}u = f, \quad (5.8)$$

where $(1 - \epsilon)\mathcal{A}^\infty u(x) + \epsilon \tilde{\mathcal{A}}u(x) = \frac{1}{2} \text{Tr}([(1 - \epsilon)Q^\infty + \epsilon Q(x)]D^2 u(x)) + \langle F(x) + Ax, Du(x) \rangle$.

Using the previous a priori estimates, it is straightforward to verify that the set of all ϵ 's such that (5.8) is uniquely solvable in $\mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$ is non-empty, closed and open in $[0, 1]$. Taking $\epsilon = 1$ in (5.8) one finishes the proof. \blacksquare

In order to state and prove Schauder estimates for the parabolic Cauchy problem involving $\tilde{\mathcal{A}}$, we define the space $\mathcal{C}_{T,d}^\gamma$, $\gamma \in (0, 3)$ non-integer. This consists of all continuous functions $v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $v(t, \cdot) \in \mathcal{C}_d^\gamma(\mathbb{R}^n)$, $t \in [0, T]$, and moreover $\sup_{t \in [0, T]} \|v(t, \cdot)\|_{\mathcal{C}_d^\gamma(\mathbb{R}^n)} < +\infty$. $\mathcal{C}_{T,d}^\gamma$ is a Banach space endowed with the norm $\|\cdot\|_{\gamma, T, d}$,

$$\|v\|_{\gamma, T, d} = \sup_{t \in [0, T]} \|v(t, \cdot)\|_{\mathcal{C}_d^\gamma(\mathbb{R}^n)}, \quad v \in \mathcal{C}_{T,d}^\gamma.$$

A function $v \in \mathcal{C}_{T,d}^{2+\theta}$, $\theta \in (0, 1)$, solves the Cauchy problem (4.2) involving $\tilde{\mathcal{A}}$ if $v(0, x) = g(x)$, $x \in \mathbb{R}^n$, and, for any $\phi \in C_0^\infty(\mathbb{R}^n)$, the real mapping: $t \mapsto \int_{\mathbb{R}^n} v(t, x) \phi(x) dx$ is continuously differentiable on $[0, T]$ and verifies, for any $t \in [0, T]$ (see (5.7)),

$$\frac{d}{dt} \left(\int_{\mathbb{R}^n} v(t, x) \phi(x) dx \right) = \int_{\mathbb{R}^n} \mathcal{A}_1 v(t, x) \phi(x) dx + \int_{\mathbb{R}^n} v(t, x) \mathcal{A}_2^* \phi(x) dx + \int_{\mathbb{R}^n} H(t, x) \phi(x) dx. \quad (5.9)$$

Theorem 5.3. *Consider the operator $\tilde{\mathcal{A}}$ in (5.1) under Hypothesis 5.1. Let $T > 0$, $g \in \mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$ and $H \in \mathcal{C}_{T,d}^\theta$. Then there exists a unique solution $v \in \mathcal{C}_{T,d}^{2+\theta}$ to the Cauchy problem (4.2) involving $\tilde{\mathcal{A}}$. Moreover, the partial derivatives $D_i v$ and $D_{ij}^2 v$ are continuous on $[0, T] \times \mathbb{R}^n$, for $i, j \in I_0$, and there exists $c > 0$, independent of g , H and v , such that*

$$\|v\|_{2+\theta, T, d} \leq c (\|g\|_{2+\theta, d} + \|H\|_{\theta, T, d}). \quad (5.10)$$

Proof. The proof is similar to the one of Theorem 5.2. Let $v \in \mathcal{C}_{T,d}^{2+\theta}$ be a solution. One first proves the following maximum principle

$$\sup_{t \in [0, T], x \in \mathbb{R}^n} |v(t, x)| \leq T \sup_{t \in [0, T], x \in \mathbb{R}^n} |H(t, x)| + \|g\|_0.$$

arguing as in (4.7) (using that $\tilde{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2$ as in the proof of Theorem 5.2).

Concerning the localization procedure which gives the required a priori estimates, we only note that, for any $\eta \in C_0^\infty(\mathbb{R}^n)$, according to (5.9), the function $v\eta$ solves

$$\begin{cases} \partial_t(v\eta)(t, x) = \tilde{\mathcal{A}}(\eta v)(t, x) - v(t, x)\tilde{\mathcal{A}}\eta(x) - \langle Q(x)D\eta(x), Dv(t, x) \rangle + H(t, x)\eta(x), & t \in (0, T], \\ (\eta v)(0, x) = \eta(x)g(x), & x \in \mathbb{R}^n. \end{cases}$$

Finally the continuity method of Theorem 5.2 works also in this case, replacing the space $\mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$ with $\mathcal{C}_{T,d}^{2+\theta}$ and gives the assertion. \blacksquare

Remark 5.4. One can weaken the assumption (ii) in Hypothesis 5.1 about F in order to prove elliptic and parabolic Schauder estimates for $\tilde{\mathcal{A}}$. To this purpose we can consider $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $F(x) = (F_1(x), \dots, F_{\tilde{p}}(x), 0, \dots, 0)$, $x \in \mathbb{R}^n$, and moreover there exist $\theta \in (0, 1)$ and $M > 0$ such that, for any $x, y \in \mathbb{R}^n$, if $|y| \leq 1$ then we have

$$|F(x) - F(x + y)| \leq M \|y\|^\theta. \quad (5.11)$$

We briefly explain how to prove elliptic Schauder estimates for $\tilde{\mathcal{A}}$ when F satisfies the previous assumptions (in a similar way one gets parabolic Schauder estimates for $\tilde{\mathcal{A}}$). First we deal with the maximum principle. Let $u \in \mathcal{C}_d^{2+\theta}(\mathbb{R}^n)$ be a solution. We consider $u_m = u * \rho_m$, where (ρ_m) are mollifiers. Under the new assumptions on F one can only show that $\tilde{\mathcal{A}}u_m$ converges to $\tilde{\mathcal{A}}u$ uniformly on compact sets of \mathbb{R}^n (compare with the proof of Theorem 5.2). This fact allows to prove that if x_0 is a local maximum for u then $\tilde{\mathcal{A}}u(x_0) \leq 0$ (see the proof of [18, Proposition 3.1.10]). Adapting the proof of [19, Proposition 2.2] one obtains the maximum principle. Then, in order to get Schauder estimates, one writes

$$\lambda u(x) - \frac{1}{2} \text{Tr} (Q(x)D^2u(x)) - \langle Ax + (F * \rho)(x), Du(x) \rangle = f + \langle F(x) - (F * \rho)(x), Du(x) \rangle,$$

where $F * \rho$ is the convolution between F and a function $\rho \in C_0^\infty(\mathbb{R}^n)$, $\|\rho\|_{L^1(\mathbb{R}^n)} = 1$, $0 \leq \rho \leq 1$, $\rho(x) = \rho(-x)$, $x \in \mathbb{R}^n$, and $\rho(x) = 0$ if $|x| \geq 1$. Using that $D_i(F * \rho)(x) = \int_{\mathbb{R}^n} (F(x - y) - F(x))D_i\rho(y)dy$ and similar formulae for higher partial derivatives, we see that $F * \rho$ satisfies (iii) in Hypothesis 1.1. Moreover by (5.11) one checks that $F - (F * \rho)$ belongs to $\mathcal{C}_d^\theta(\mathbb{R}^n)$. Finally, straightforward computations allow to get Schauder estimates for $\tilde{\mathcal{A}}$.

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Dipartimento di Matematica, Università di Torino,
 via Carlo Alberto 10, 10123, Torino, Italy.
 enrico.priola@unito.it