TIME IRREGULARITY OF GENERALIZED ORNSTEIN–UHLENBECK PROCESSES

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Abstract. The paper is concerned with the properties of solutions to linear evolution equation perturbed by cylindrical Lévy processes. It turns out that solutions, under rather weak requirements, do not have càdlàg modification. Some natural open questions are also stated.

1. Introduction

In the study of spdes with Lévy noise a special role is played by linear stochastic equations:

$$\left\{\begin{array}{rcl} dX(t) &=& AX(t)\,dt + dL(t),\ t\geq 0,\\ X(0) &=& 0\in H,\end{array}\right. \tag{1.1}$$

on a Hilbert space H. In (1.1) A stands for an infinitesimal generator of a C_0-semigroup $S(t)$ on H and L is a Lévy process with values in a Hilbert space U often different and larger than H. The weak solution to (1.1) is of the form, see e.g. [6],

$$X(t) = \int_0^t S(t-s)dL(s),\ t\geq 0. \tag{1.2}$$

The time regularity of the process X is of prime interest in the study of non-linear stochastic PDEs, see e.g. [8]. If the Lévy process L takes values in H than the solution X has H-càdlàg trajectories because of the maximal inequalities for stochastic convolutions due to Kotelenez [5], see also [6]. However the process X can take values in H even if the space U is larger than H and L does not evolve in H. It turns out that if L is the so called Lévy white noise the process X does not have $H-$ càdlàg trajectories, see [2] and [6], although it may have a version taking values in a subspace of H of rather regular elements. The main reason for this phenomenon was the fact that the process L had jumps not belonging to the space H. It was therefore natural to conjecture that if the jumps of the process L belong to H than

Date: November 12, 2009.

Key words and phrases. Lévy white noise, subordinator process, Langevin equation, space regularity, stochastic heat equation, non-càdlàg trajectories, stochastic Burgers equation.

Supported by the Polish Ministry of Science and Education project 1PO 3A 034 29 "Stochastic evolution equations with Lévy noise" and by EC FP6 Marie Curie ToK programme SPADE2.

the càdlàg modification of X should exist. The present paper shows that this is not always the case, and that in fact the problem of characterizing equations (1.1) which solutions have càdlàg modification is still open. This is true even in the diagonal case for which, in the Gaussian case, there are satisfactory answers, see [4], [3].

In the present paper we consider a class of processes L which have expansion of the form

$$L(t) = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n, \; t \geq 0. \tag{1.3}$$

where L^n are independent, identically distributed, càdlàg, real valued Lévy processes with the jump intensity ν not identically 0. Here (e_n) is an orthonormal basis in H and β_n is a sequence of positive numbers. It is not difficult to see that the jumps of the process L belong to H but only under special assumptions the process L evolves in H. We show that, in general, the process X does not have an $H-$ càdlàg modification. The case of stochastic heat equation will be considered with some detail.

2. Main theorem

The main result of the paper is the following theorem

Theorem 2.1. *Assume that the process X in (1.2) is an H-valued process and that the elements of the basis (e_n) belong to the domain $D(A^*)$ of the operator A^* adjoint to A. If β_n do not converge to 0, then, with probability 1, trajectories of X have no point $t \in [0, +\infty)$ in which there exists the left limit $X(t-) \in H$ or the right limit $X(t+) \in H$.*

Corollary 2.1. *Assume that the hypotheses of Theorem 2.1 hold. Then the process X has no an H–càdlàg modification.*

Remark 2.2. Consider an important case when the operator A is self-adjoint with eigenvectors e_n and the corresponding eigenvalues $-\lambda_n < 0, \; n = 1, 2, \ldots$ tending to $-\infty$. Denote by X^n the $\mathbb{R}$-valued Ornstein–Uhlenbeck process defined by

$$\left\{ \begin{array}{rcl} dX^n(t) & = & -\lambda_n X^n(t)\,dt + \beta_n dL^n(t), \; t \geq 0, \\ X^n(0) & = & 0, \end{array} \right. \tag{2.1}$$

and identify H with l^2. The regularity of the process $X(t) = (X^n(t))$ with L^n independent Wiener processes was considered in the paper [4] were conditions, close to necessary and sufficient, for continuity of trajectories, were given. If the processes $\left(L^n\right)$, $n \in \mathbb{N}$, are without gaussian part, i.e. of pure jump type, and ν is a symmetric measure, then the necessary and sufficient conditions for the process $(X^n(t))$ to take values in l^2 are given in a recent paper [7].

The proof of the theorem will be a consequence of two lemmas. The first of them is a variant of the well-known Cauchy criterium for the existence of limit.

Lemma 2.3. *A function* $f\colon [0,+\infty) \to E$*, where* E *is a Banach space (with norm denoted by* $\|\cdot\|$*) admits left limit at some* $t>0$ *(respec. right limit at some* $s\ge 0$*) if and only if for an arbitrary* $\varepsilon>0$ *there exists* $\delta>0$ *such that*

$$\mathrm{osc}_f((t-\delta,t))<\varepsilon \quad (\textit{respec.}\ \ \mathrm{osc}_f((s,s+\delta))<\varepsilon), \tag{2.2}$$

where, for $\Gamma\subset[0,1]$*,*

$$\mathrm{osc}_f(\Gamma):=\sup_{s,t\in\Gamma}\|f(t)-f(s)\|.$$

Lemma 2.4. *Assume that for some* $0<r_1<\infty$*,* $\nu((-\infty,-r_1]\cup[r_1,\infty))>0$*. Let* τ_n *denote the first jump of the process* L_n *of magnitude at least* r_1*, in particular*

$$|\Delta L^n(\tau_n)|\ge r_1.$$

Then, with probability 1*, the set*

$$\{\tau_n\colon n\in\mathbb{N}^*\}$$

is dense in the interval $(0,+\infty)$*.*

Proof. Recall that τ_n, $n\in\mathbb{N}^*$, are independent and exponentially distributed with parameter $\lambda=\nu(\{t\in\mathbb{R}\colon |t|\ge r_1\})$ (independent of n). Let $\alpha,\beta\in\mathbb{Q}$ be such that $0<\alpha<\beta$ and let

$$A_n:=\{\tau_n\in(\alpha,\beta)\}.$$

Then, for any $n\ge 1$, $\mathbb{P}(A_n)=\mathbb{P}(A_1)\in(0,1)$, A_n are independent events and $\sum_{n=1}^\infty\mathbb{P}(A_n)=\infty$. Consequently, by the second Borel–Cantelli Lemma, with probability 1, there exists $n\in\mathbb{N}^*$, such that $\tau_n\in(\alpha,\beta)$. Since the family

$$\{(\alpha,\beta)\colon \alpha,\beta\in\mathbb{Q},\ 0<\alpha<\beta\}$$

is countable, the result follows. □

Proof of Theorem 2.1. Since X is a weak solution, see e.g. [6], for each n,

$$d\langle X(t),e_n\rangle_H=\langle X(t),A^*e_n\rangle_H dt+\beta_n dL^n(t). \tag{2.3}$$

Denote the processes $\langle X(t),e_n\rangle_H$ by $X^n(t)$.

Passing to subsequences we can assume that for some $r_2>0$ and for all n, $\beta_n\ge r_2$. Let τ_n denote the moment of the first jump of the process L_n of the absolute size greater than or equal r_1. These numbers form, with probability 1 (say, for any $\omega\in\Omega_0$) a dense subset of the interval $(0,+\infty)$ (see Lemma 2.4), and, at each moment τ_n, the process β_nL_n has a jump of the absolute size at least r_1r_2.

Arguing by contradiction, let us assume that there exists $\omega \in \Omega_0$ such that at some time t_ω the left limit or the right limit of $X(\cdot, \omega)$ exists. Let us assume that $t_\omega \geq 0$ and that there exists the right limit at t_ω. By Lemma 2.3 this means that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$osc_{X(\cdot,\omega)}((t_\omega, t_\omega + \delta)) < \varepsilon. \tag{2.4}$$

Let $\varepsilon > 0$ be any number smaller than $r_1 r_2$ and let $\delta > 0$ be such that (2.4) holds. There exists a natural number n_0 (depending also on ω) such that

$$\tau_{n_0}(\omega) \in (t_\omega, t_\omega + \delta).$$

Note that $\|X(t,\omega) - X(s,\omega)\| \geq |X^n(t,\omega) - X^n(s,\omega)|$, for any $n \geq 1$, $t, s \geq 0$. Using also equation (2.3), we infer

$$\liminf_{s \nearrow \tau_{n_0}(\omega)} \|X(\tau_{n_0}(\omega), \omega) - X(s,\omega)\| \geq \liminf_{s \nearrow \tau_{n_0}(\omega)} |X^{n_0}(\tau_{n_0}(\omega), \omega) - X^{n_0}(s,\omega)|$$

$$= \liminf_{s \nearrow \tau_{n_0}(\omega)} |\beta_{n_0} L_{n_0}(\tau_{n_0}(\omega), \omega) - \beta_{n_0} L_{n_0}(s,\omega)| \geq r_1 r_2 > \varepsilon,$$

which contradicts the statement (2.4). □

The result raises some natural questions.

Question 1. Does the assumption that the sequence (β_n) tends to zero imply existence of a càdlàg modification of X?

Question 2. Does the assumption that $e_n \in D(A^*)$ is essential for the validity of the Theorem 2.1?

Question 3. Is the requirement that the process L evolves in H also necessary for the existence of H- càdlàg modification of X?

3. Heat equation with α–stable noise

In the present section we assume that $A = \Delta$ is the Laplace operator with the Dirichlet boundary conditions on $\mathcal{O} = (0, \pi)$. Let $H = L^2(\mathcal{O})$ and

$$D(A) = H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}), \quad Au = \Delta u, \ u \in D(A). \tag{3.1}$$

It is well known that A is a self-adjoint negative operator on H and that A^{-1} is compact. Hence A is of diagonal type, with respect to eigenfunctions $(e_j)_{j=1}^\infty$, where

$$e_j(\xi) = \sqrt{\frac{2}{\pi}} \sin(j\xi), \quad \xi \in \mathcal{O}, \ j \in \mathbb{N}^*.$$

The corresponding eigenvalues of the operator $-A$ are

$$\lambda_j = j^2,\ j \in \mathbb{N}^*.$$

Setting $\beta_j = 1$, $j \in \mathbb{N}$, and assuming that L^n are independent, identically distributed, càdlàg, real valued $\alpha-$ stable Lévy processes, $\alpha \in (0,2]$, we will work with a "white" $\alpha-$ stable process:

$$L(t) = \sum_{j=1} L^j(t) e_j,\ t \geq 0,$$

and with the solution X of

$$dX(t) = \Delta X(t) + dL(t),\ t \geq 0,\quad X(0) = 0. \tag{3.2}$$

For $\delta \geq 0$, define $H_\delta = D(A^{\delta/2})$ with the naturally defined scalar product. Then in particular $H_0 = H$, $H_1 = H_0^1(\mathcal{O})$ and $H_2 = D(A)$ and moreover,

$$H_\delta = \Big\{ x \in H \colon \sum_{j=1}^{\infty} \lambda_j^\delta |x_j|^2 < \infty \Big\}$$

where $x_j := \langle x, e_j \rangle$, $j \in \mathbb{N}^*$. For $\delta < 0$, by H_δ we denote the extrapolation space which can be defined as $D(A^{-\delta/2})$, or more precisely as the completion of the space H with respect to the norm $|x|_\delta := |A^{-\delta/2} x|$, $x \in H$. The Hilbert space H_δ can then be isometrically identified with the weighted space l_δ^2,

$$l_\delta^2 = \Big\{ x = (x_j) \colon \sum_{j=1}^{\infty} \lambda_j^\delta |x_j|^2 < \infty \Big\},$$

equipped with the norm $|x|_\delta := (\sum_{j=1}^\infty \lambda_j^\delta |x_j|^2)^{1/2}$.

Proposition 3.1. *Assume that X solves equation (3.2) and $\alpha \in (0,2)$. Then:*
i) The process L is $H_\delta-$ valued, and thus H_δ–càdlàg, if and only if $\delta < -1/\alpha$.
ii) The process X is H_δ–valued if and only if $\delta < 1/\alpha$.
iii) If $\delta < -1/\alpha$ then the process X is H_δ–càdlàg.
iv) If $\delta \geq 0$ then the process X has no H_δ –càdlàg modification.

Proof. *i*) It follows from [8, Proposition 3.3] that the process L takes values in the space H_δ if and only if

$$\sum |\lambda_j^{\delta/2}|^\alpha < \infty.$$

Since $\lambda_j = j^2$, $|\lambda_j^{\delta/2}|^\alpha = j^{\delta\alpha}$, we infer that the process L takes values in the space H_δ if and only if $\delta < -1/\alpha$.

ii) The argument from the proof of *i*) applies.

iii) By the maximal inequalities for stochastic convolution, [5], we infer that if the procees L is H_δ–valued then the process X is H_δ-càdlàg.

iv) This is a direct consequence of our Theorem 2.1. □

It is of interest to compare the stable case $\alpha \in (0, 2)$ with the Gaussian case $\alpha = 2$.

Proposition 3.2. *Assume that X solves equation (3.2) and $\alpha = 2$. Then*
i) The process L is H_δ– valued, and thus H_δ–continuous, if and only if $\delta < -1/2$.
ii)The process X is H_δ– valued if and only if $\delta < 1/2$.
iii)The process is X is H_δ– continuous if and only if $\delta < 1/2$.

Proof. This is a well known result. Parts *i*) and *ii*) can be proved in the same way as in the previous theorem. To prove *iii*) it is enough to apply a sufficient condition for continuity from [3], namely that $\exists \beta > 0, \exists T > 0$ such that

$$\int_0^T t^{-\beta} \|e^{t\Delta}\|^2_{L_{HS}(H_0,H_\delta)} dt < +\infty,$$

where $\|\cdot\|_{L_{HS}(H_0,H_\delta)}$ denotes the Hilbert–Schmidt norm of an operator from H_0 into H_δ. One can also use [4]. □

We see that the regularity result for the Gaussian Ornstein–Uhlenbeck process does not have a precise analog for the α–stable process. We have the following natural open question where our Theorem 2.1 is here not applicable.

Question 4. Is the process X from Proposition 3.1, H_δ-càdlàg for $\delta \in [-1/\alpha, 0)$?

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