DAVID SLAVIT

AN ALTERNATE ROUTE TO THE REIFICATION OF FUNCTION

ABSTRACT. This paper presents an alternate perspective for utilizing the action/process/object framework when discussing student development of conceptions of function. After a review of related theories, a property-oriented view of function is described which is based on visual aspects of functional growth. The theory is supported with data on student learning. The property-oriented view of function incorporates and extends previously described frameworks used in analyzing functional understandings, including the covariance and correspondence views (Confrey & Smith, 1991; Thompson, 1994). The property-oriented view differs from the covariance view in that less emphasis is placed on the manner in which the variables are changing and more emphasis is placed on the properties that result from these changes. The property-oriented view differs from a correspondence view in that functional properties such as invertibility and domain give rise to a different kind of thinking about functions than do properties such as symmetry, linearity, continuity, etc. Implications for further research and curriculum development are also provided.

1. INTRODUCTION

> All things can be systematically arranged if only one has within one's self the idea of order and sequence. One cannot have the idea of order, however, until one has symbols and some kind of an established sequence of thought. I confess I am astonished when I see some of the analyses which purport to be the scientific foundations on which school curriculums are to be built and find no mention of these general ideas of order and arrangement and precision. I am told that the school should teach children how to make change and how to measure wall paper and how to tell time and that sections of arithmetic should be devoted to these specific tasks, but I look in vain for any appreciation of the fact that the school ought to lead pupils who have only a hazy and unsystematic notion of the world to see the value of arrangement and order in all thinking and to cultivate the general ideas of regularity and precision. (Charles Judd, 1928)

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A very old thought problem is the careful discrimination between a bowl and a cup. One way of handling this situation is to give careful consideration to the physical properties which are present in a respective prototype. What gives a bowl its “bowlness” and what gives a cup its “cupness”? In essence, this approach to the problem discerns physical properties from various examples of cups and bowls in an effort to both describe and discriminate the abstract cases. One uses physical properties from previously experienced, tangible objects to construct and describe properties of, in essence, mental objects.

Such generalizations embody the essence of mathematical thought and development. Recently, the theory of reification (Sfard, 1989, 1992) set forth the notion that students come to develop more permanent understandings (abstract objects) through the enrichment of understandings of associated mathematical processes and actions; the abstract objects are the embodiments of these actions (see also Gray & Tall, 1994, for a discussion of procept). The following discussion involves an alternate approach of operationalizing these theoretical notions into algebra classrooms that focus on functions and make use of graphs and graphing technologies. It is important to acknowledge the setting in the development of the theory, because algebra classrooms with other foci and vehicles of presentation may spur conceptual development of another kind (Briedenbach et al., 1992). The theory suggests that students, in part, can conceive of functions as entities possessing various growth properties of a local and global nature. These properties include intercepts, symmetry, asymptotes, and other types of behaviors. Students can then conceive of functions as abstract objects either possessing or not possessing these previously experienced properties.

Although discussions of a property-oriented view of function would certainly be viable in regard to a relational view of function, this paper focuses more on functional understandings involving growth and covariance. This is mainly due to the instructional nature of the observed classrooms and the emphasis on functional growth properties often associated with classrooms that make use of graphing technologies. Certain functional properties, such as invertibility and onto, are relational in nature; these properties involve analyzing the nature of the relationship between inputs and outputs on a “point-to-point” basis rather than in regard to the manner in which the relationship between inputs and outputs is changing. It is possible to outline a property-oriented view of function based on these constructs. Although a discussion of a relational property-oriented view would be both interesting and relevant to the educational enterprise, it is beyond the scope of this paper. For the purposes of this paper, the property-oriented view will only refer to student conceptions of function relative to functional growth and covariance.

The theory of a property-oriented view of function suggests that students come to understand the concept of function by transforming their experientially-based perceptual patterns of functional growth behaviors into well-formed understandings of specific functional attributes. We will be begin by analyzing current approaches to the discussion of student understandings of function, and then the property-oriented view will be described. Relationships between these theories along with empirical support will be provided, followed by implications for curriculum.

2. THEORIES OF STUDENT CONCEPTIONS OF FUNCTION

Most theories of student development of the function concept focus on differences between action-oriented conceptions and object-oriented conceptions. While much agreement exists in regard to the former, several different theories have been advanced to describe the latter. We begin with a quick description of an action view of function, and then discuss some of the various approaches used to describe a more object-oriented view of function.

2.1. *Action Views of Function*

Evidence exists that students initially acquire an *action* or *operational* view of function (Briedenbach et al., 1992; Sfard, 1989). In the context of an action/process/object theory of conceptual development, an action view involves an understanding of function as a non-permanent construct. An action view pertains to the computational aspects associated with functions, such as an arithmetic process or a ‘function machine’. For example, one can consider the function $f(x) = 3x^2 - 7$ to be an algorithm used to compute numeric values for a given input. This conception does not require an awareness of patterns and regularities that may exist between the numeric values of successive inputs and outputs, nor attention to causal and dependency relationships between inputs and outputs. An action conception is concerned with the computation of a single quantity for a single numeric value via a given algorithm or rule of association.

2.2. *Object-Oriented Views of Function*

Students can extend their action-oriented understandings to incorporate more permanent, object-oriented notions into their concept image (Vinner, 1983). A discussion of an object-oriented view of function will begin by reexamining several existing theories. Most of these theories involve conceptions of function which have been referred to as a *correspondence* or *covariance* view. Confrey & Smith (1991, 1995) and Thompson (1994) have previously provided frameworks for distinguishing these views of function. An alternate theory for an object-oriented view of function will then be presented that, while borrowing heavily from previous ideas, discusses student development in a different context.

2.2.1. *Correspondence (relational) view of function*

Understandings based on causal and dependency relationships between input-output pairs comprise the essence of a relational view. These relationships can exist between individual inputs and outputs (Piaget et al., 1977; Greeno, 1988) or generalize to an entire set of input-output pairs (Briedenbach et al., 1992). Textbooks and school curricula have traditionally given emphasis to this view of function, particularly in the United States (Confrey & Smith, 1995; Thompson, 1994).

Briedenbach et al. (1992) provide evidence to suggest that instruction which utilizes computer programming can promote relational conceptions. They define a *process* conception of function as a complete understanding of a given transformational activity performed on a function, consisting of causal and dependency relationships which exist between the dependent and independent variables. Briedenbach et al. claim that the process conception provides an entryway into an object-oriented understanding of function. Students are more able to comprehend properties such as 1–1, onto, and invertibility once a process conception is achieved.

Sfard (1991, Sfard & Linchevski, 1994) describes an object-oriented conception as the *reification* of an action view. For example, one can consider the expression $3(x+5)+1$ as an algorithm with which to produce various outputs. However, one can also “see” the expression as a certain number in its right; the expression becomes the result of the process. Student difficulties in simultaneously comprehending these meanings of an expression have been referred to as the ‘process-product dilemma’ (Davis, 1975). Thinking of this type illustrates the beginnings of an object-oriented view since the expression is considered to be a fixed value of an unknown. Hence, a single input-output action is conceived as a single entity. Generalizing this conception to involve the notion of variable, where the above expression simultaneously denotes several processes, represents thinking in line with functional algebra and is considered to be a structural conception. Sfard’s emphasis is on the development of ‘abstract objects’ as a product of a deeper understanding of a mathematical operation (Sfard & Linchevski, 1994; Sfard & Thompson, 1994).

2.2.2. *Covariance view of function*

A distinct, but related, view of function beyond an action conception involves an understanding of the manner in which the dependent and independent variables change. Analyzing, manipulating, and comprehending the relationships between changing quantities illustrate the covariance view. For example, given the linear equation $y = 2x + 3$, Confrey & Smith (1995) have investigated the manner in which students establish relationships such as ‘twice x plus three’ and ‘doubling’ (the latter is a special case of the operation of ‘splitting’).

Other investigations of covariance have been focused more on properties of functional growth. For example, Monk & Nemirovsky (1994) describe the investigations of Dan, a twelfth grade student exposed to an air flow device used to produce flow rate vs. time and volume vs. time relationships. After producing graphic descriptions of the situations, Dan begins to analyze the functional situation:

> Not only does Dan seek to ground these visual attributes in his experience of the device, but he also seeks to move in the direction of abstraction, of using these attributes as the terms, or tokens, of a system of reasoning about the situation (p. 149).

They claim that visual properties of functions perceived in graphic settings led Dan into an analysis which utilized notions such as ‘steepness’ and ‘spread apart’. They also claim that this perspective should be utilized by other researchers to discuss what the students know about functional situations, rather than ‘fixing’ misconceptions that result from a visual analysis of functions.

Schwarz and Dreyfus have made extensive investigations of students’ acquisition of the function concept in the setting of their Triple Representation Model (TRM) (Schwarz et al., 1990; Schwarz & Dreyfus, 1995). This software allows dynamic interplay between graphic, numeric, and equation settings. They suggest that students’ perform two different types of actions on these different settings; the first type of action changes the function itself (e.g., a shift translation) while the second type of action changes the ‘representative’ of the function (e.g., rescaling a graphic image on a graphing calculator). Schwarz & Dreyfus (1995) suggest that these types of actions allow students to identify invariant properties which are characteristic of the functions under investigation. The latter two perspectives are markedly different from those above due to their focus on function properties and the physical make-up of graphs of functions. In fact, the following discussion of the property-oriented view blends this perspective with that of Sfard to discuss how a student can reify the notion of function as a mathematical object that possesses or does not possess various functional properties.

3. PROPERTY-ORIENTED VIEW OF FUNCTION

3.1. *What is a 'Property'?*

Thompson (1994), in an important and informative discussion of research on mathematical functions, argues that the notion of 'representation' is in need of explication. Further, he states that what are normally termed as 'multiple representations of function' should instead be discussed in more subjective terms; multiple function representations do not represent functions, but rather they provide a means of making connections and allow us to establish a sense of invariance.

It is in this context that I discuss the idea of 'property'. Given any physical object, one can describe various features that it possesses. Unless a process to change the object is performed, or until new insights are experienced, these properties will remain invariant to the observer. Hence, defining the notion of property in the context of a single object and a single observer is manageable. The difficulty arises when one must define property in the context of a more general idea, such as one that encompasses numerous and varied types of objects and entails an abstract character (e.g., a mathematical function). Here is where the notion of invariance is critical. One looks for characteristics that are invariant across given examples and then classifies subgroups based on their possession of an observable characteristic. For example, we can discuss what linear functions are by describing all of the properties that linear functions must possess. In other words, linear functions can be described by noting those aspects which remain invariant across every example of a linear function. In reality, anomalous cases, such as the fact that a linear function of the form $f(x) = k$ $(k \neq 0)$ has no intercepts, lead us to make use of 'most' of the examples of a particular object in establishing an image based on properties. Certain types of functions can also be described by the properties that they do not possess, such as discontinuity or a lack of symmetry (although these 'negative' traits could also be labeled as properties).

When discussing properties of the general concept of function, one must consider all of the characteristics associated with an object satisfying the definition of function. This is clearly an exacting task, particularly since different individuals may notice different properties of any given functional situation. Further, the varied nature of functional properties allows us to classify them into different types. As seen below, certain global properties involve an analysis of the entire function, while local properties are involved with individual or selected input-output pairs. Global functional properties include symmetry and periodicity, and local properties include intercepts and points of inflection. Certain properties, such as continuity, Common Functional Properties

Global				Local
Growth				
Periodicity				Points of Inflection
Symmetry		Slope	Concavity	
Monotonicity				Cusps
Horizontal and Slant Asymptotes				Vertical Asymptotes
		Contexts	Extrema	
	Boundedness	Continuity		
		Multivariable		
Constant		Sign		Intercepts
		Differentiability		
	Integrability			
	Invertibility			
One-to-one		Domain		
Onto		Range		Kernel
Correspondence				

can transcend these classifications. The classification given above also suggests that functional properties can deal more with aspects of growth and covariance than with the relational aspects of function. Of course, different classifications of these properties would certainly be viable. For example, one could argue that ‘boundedness’ has a dominant local interpretation, or that ‘periodicity’ has more to do with correspondence than with functional growth behavior. This classification is simply one way of describing various functional properties in the context of student thought and application. Hence, we must keep in mind the complex and varied nature of functional properties when attempting to describe student understandings based on this notion. But there is precedence for this perspective beyond discussions of functional understanding. Kieren & Pirie (1991; Kieren, 1990, 1994) have used ‘property noticing’ as a major part of a general theoretical framework for describing mathematical understanding.

One final remark is needed. The above discussion focused solely on properties of functions when thought of as mathematical objects. But we must always remember the lack of clarity involved in asserting that anyone has an object-oriented understanding of any mathematical idea (Thompson, 1994; Schoenfeld et al., 1993). Moreover, processes performed on functions, such as changing a viewing window, factoring, and looking for patterns in data, all have clearly demarcated properties of their own[2] (Schwarz & Dreyfus, 1995). In addition, there are properties of the general action view of function, such as computational properties, causation, and dependence. The following section attempts to address the specific issue of describing understandings of function as a mathematical object in regard to properties of functional growth.

3.2. *Property-Oriented View of Function*

A property-oriented view of function deals with the gradual awareness of specific functional growth properties of a local and global nature, followed by the ability to recognize and analyze functions by identifying the presence or absence of these growth properties. Through experiences with function exemplars, students may develop a concept image of function as a related set of procedures and functional properties in a variety of notational systems. In essence, the procedures performed on functions give rise to an understanding of functional growth properties such as zeroes, concavity, and asymptotes. Particular contexts elicited by the semantic domain of the functional situation can also enrich the meanings in the conceived functional properties. The growth properties can be specific to a function class, such as linear slope, or generalize to several function classes, such as symmetry. As these understandings are built, a student can reify the notion of function as a mathematical object capable of possessing or not possessing these properties (Slavit, 1994a, 1994b). Hence, asserting that someone ‘has’ a property-oriented view of function is not completely unambiguous. One must decide if the student has developed a sufficient awareness of a sufficient number of functional properties, and then determine if the student can utilize this information to form an adequate conception of function as an abstract mathematical object capable of possessing these properties. Because investigations of student conceptions of functions is, to say the least, an extremely complex task, we should always remember the comments of Schoenfeld et al. (1993):

> Saying when a student actually ‘has’ the object perspective is not a simple matter. It is not a yes/no kind of knowledge, but one of degrees, and the process of learning is not of simple monotonic growth, but one that includes a fair amount of oscillation (p. 88).

3.2.1. *Development of a property-oriented view*

Prior work in this area (e.g., Ruthven, 1990; Quesada & Maxwell, 1994; Schwarz & Dreyfus, 1995) has led me to hypothesize that the property-oriented view is established through two types of experiences. First, the property-oriented view involves an ability to realize the equivalence of procedures that are performed in different notational systems. Noting that the processes of symbolically solving $f(x) = 0$ and graphically finding x-intercepts are equivalent (in the sense of finding zeroes) demonstrates this awareness.

Second, students develop the ability to generalize procedures across different classes and types of functions. Here, students can relate procedures across notational systems, but they are also beginning to realize that some of these procedures have analogues in other types of functions. For example, one can find zeroes of both linear and quadratic polynomials (as well as many other types of functions), and this invariance is what makes the property apparent.

The previous classification of ‘common functional properties’ provides a long list of properties observable to the student. Once a student has become familiar with these and other functional growth properties through various experiences, he or she can ‘see’ a function as an object either with or without these properties and outside the context of examples of specific functions or specific function types. Hence, one makes use of a property-oriented view of function in analyzing the exact nature of specific examples of functions and non-functions. It is clear that property-oriented conceptions of particular classes and types of functions develop concurrently with a more general view of function. In fact, a property-oriented view of function could continue to develop for as long as the student was exposed to novel properties.

Just as a relational view is convenient when dealing with certain operations on functions (such as composition), a property-oriented view helps to relate specific examples of functions to their corresponding growth behaviors. For example, a quadratic function could be viewed as a continuous function with exactly one extrema, at most two zeroes, and which is symmetric about a vertical line (with, of course, second-degree growth). Students can acquire this view by analyzing several examples of quadratic and non-quadratic functions, and then discriminate amongst the growth behaviors as well as the physical characteristics of the graphs, both similar and alike. Hence, a property-oriented view of function develops through an awareness of specific properties of functional growth as well as through understandings of specific types of functions possessing these properties. Both of these developments are interdependent.

3.2.2. *Discussion*

The previous section asserted that a form of reification occurs when the student understands the property-oriented view without a reliance on specific examples of functions or function types. In other words, properties such as extrema and asymptotes are understood as abstract object themselves, so that the idea of function can itself become an abstract object able to possess these deeply understood properties. This can be manifested in a shift in the language of the student from properties of processes to properties of objects (Sfard, personal communication).

The development of a property-oriented view of function will take time since it is dependent on knowledge of several different functional properties, notational systems, and classes of functions. If linear and quadratic functions are studied almost exclusively, as is the case in many beginning algebra courses, then a student's 'library' of functional properties will be quite small. To this student, functions are certainly well-behaved and continuous, have very simple growth behaviors, are either monotonic or change their direction of growth once, and are zero at most twice (Slavit, 1994a, 1996). At this stage, a property-oriented view of function has been developed, but it is certainly limited. When other polynomial (usually up to fourth- or fifth-degree), exponential and logarithmic, trigonometric, radical, and absolute value functions are added, as is the case in many second and third year algebra courses, the student's library of functional properties will increase, and a property-oriented conception of function strengthens. By seeing examples of different types of functions that share functional properties (linear and simple radical functions sharing monotonicity, quadratic and simple absolute value functions sharing a unique extrema, quadratic and some trigonometric functions sharing symmetry), and by noting similarities and differences in the functions, the student can relate specific properties to a variety of exemplars. Generalizing these notions of properties to a general concept image of function describes the essence of the property-oriented view.

4. SYNTHESIS OF THE VIEWS

The previous views of function are not presented as disjoint avenues of student development, nor are they intended to completely describe all of the ways in which students can develop a concept image of function. Further, the above discussion of the property-oriented view is in line with characteristics of algebra courses which utilize graphs and multiple notations, as well as student responses to this instruction (Ruthven, 1990; Slavit, 1996; Monk & Nemirovsky, 1994; Quesada & Maxwell, 1994). It must also be stated that the intent of this paper is not to develop a stage theory for the development of function. Thompson (1994) articulates the complexity of such a discussion. A property-oriented conception may be the most developed view of function that a particular student constructs, but it is also

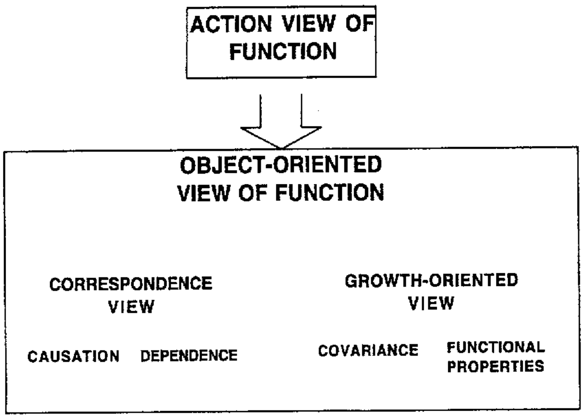

Figure 1. Components of an object-oriented view of function.

possible that it may be marginalized. Instruction is likely to influence the sequencing of this development.

The previous discussions of different theoretical perspectives on students' conceptions of function should also not be construed as a compartmentalized account. Each of these theories has strands that interrelate, and, in fact, the property-oriented view has been presented to extend and not replace these theories (Figure 1). Utilizing the covariance view in student thinking and problem solving addresses growth behavior, but it does not explicitly address the many functional properties to which students attend when analyzing functions with specific growth behaviors. The essence of a property-oriented view makes use of the notion of covariance to discern functional growth properties (periodicity, symmetry, etc.) that help define the nature of the function under investigation. Using this perspective, the development of function as an abstract object becomes, from the student's point of view, the recognition of properties exhibited by a function or functional situation that are familiar to and previously experienced by the student.

The correspondence view of function can also be discussed from the standpoint of functional properties (refer to the previous classification of 'common functional properties'). However, these functional properties have more to do with relationships between input-output pairs than with functional growth and covariance. Properties such as invertibility, domain, onto, and kernel are considered when thinking about function as a correspondence relation. A different set of properties, including symmetry, intercepts, extrema, and cusps, are considered when thinking about a function's growth behavior. Other properties, such as continuity, are harder to classify. Hence, the covariance and correspondence views of function, under the property-oriented framework, should not be considered as contrasting or distinct viewpoints, but rather can be considered complementary ways of thinking about the concept of function as a mathematical object possessing various properties.

Students who acquire any of the object-oriented views of function can investigate functional contexts without a reliance on specific algorithms. Instead, emphasis can be placed on global behaviors such as growth rate, on relationships between specific local properties, or on the overall relationships between inputs and outputs that help define the global properties of a given function. An object-oriented view is helpful in establishing a proceptual understanding of functional notations (Gray & Tall, 1994), an understanding that transcends the action/object duality. Students who see functional notations proceptually have the flexibility to think about function as an action, an object, or both. Further flexibility arises when the student incorporates different perspectives into his or her concept image, such as the above correspondence and covariance views. These views also allow the student to better understand actions performed on a function, such as a 'shift' translation (e.g., changing $f(x)$ to $f(x+3)$) or taking a derivative. It would be quite difficult for a student to completely understand an action he or she performed on a function if an object-oriented view of function was not yet achieved.

5. EMPIRICAL SUPPORT

Data on the relational and covariance views of function have been previously cited and discussed. This section discusses an empirical basis for setting forth a theory incorporating a property-oriented perspective. A brief overview of research will be followed by a more detailed account of a particular study conducted by the author.

5.1. *Introduction*

There is evidence that students obtain a property-oriented view of function, particularly when exposed to instruction that makes uses of graphs and graphing technologies. Ruthven (1990) found that students who used graphing calculators were better able to describe a given graph in symbolic terms than those who did not use the calculators, and that their ability to do this relied heavily on their knowledge of functional properties. These students were also better able to identify and distinguish between classes of functions. At the Technology Educational Resource Center (TERC), children have been allowed to apply their own intuitive functional understandings to familiar situations, mathematize them using invented representations, and then apply formal notations to the situation at both the elementary (Tierney & Nemirovsky, 1995) and secondary levels (Nemirovsky & Rubin, 1992; Monk & Nemirovsky, 1994). Their results suggest that students make extensive use of properties in analyzing functional situations, particularly in regard to the use of graphic notations. Magidson (1992), describing a teaching experiment in prealgebra with a radical constructivist framework, argues that computer environments can naturally foster student development of properties, such as slope, if the appropriate learning environment is established.

Several dissertations were conducted over the last few years that also lend support to the property-oriented theory. Rich (1990) found that students who used graphing calculators attended more to global functional properties than a control group, especially when in a graphic domain. An interesting study by Teles (1989) investigated the effect of a curriculum that placed heavy emphasis on functional properties in the instructional process. Properties included symmetry, periodicity, rates of change, and asymptotes. Her results suggest that such a curricula can develop student understandings of functional properties that lead to success on abstract algebraic tasks. However, no study was found that explicitly investigated the development of an object-oriented view of function from the property-oriented perspective.

5.2. *Focus Study*

A year-long study (Slavit, 1994a, 1996) conducted in a high school Honors Algebra II course which made extensive use of graphing calculators provides detailed support for the theory of a property-oriented view of function. Questionnaires, test items, and case study interview data indicated that some students reified certain types of functions using property-oriented notions, but no evidence was found to suggest that the general concept of function was reified in this manner. One way that students developed a property-oriented view was to focus on the type of function or functional situation (linear, exponential, etc.) and relate this to its various graphic, numeric, or symbolic features. Properties of function types commonly discussed in the course included symmetry, continuity, zeroes, extrema, and end growth behaviors. Many of these properties appeared in students' descriptions of functions at the end of the year:

Curt: Can have x, x^2, x^3, x^4, etc. Can go from neg-pos or pos-neg or neg-neg or pos-pos (Author's Note: These notations represent the various shapes of polynomials, such

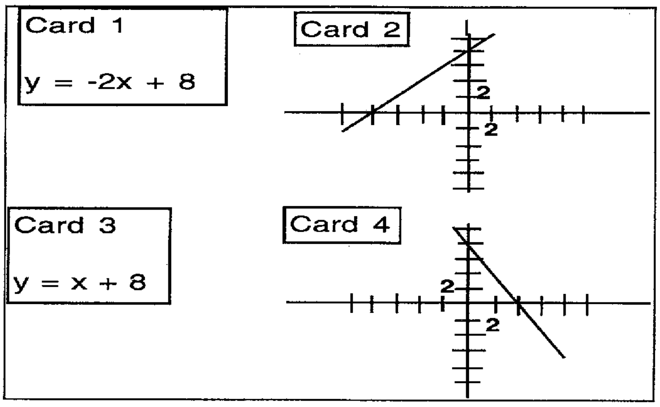

Figure 2. Translation task.

as starting 'at' negative infinity and eventually approaching positive infinity as x approaches positive infinity, etc.). Can be a parabola.

DB: They can be linear, exponential, logarithmic, etc. All must have only one output value for each input value. The graphs can look like anything, but you must be able to draw a horizontal line anywhere and intersect the function only once.

Joe: Slopes, vertices, zeroes, extraneous roots, asymptotes.

Marilyn: Can move up and down; to the left and right, and become steeper or wider.

Cathy: They can cross the x axis a certain number of times, be a straight line or parabola and others, etc.

The majority of the property-oriented responses involved verbal or graphic descriptions of the possible growth rates of functions, local behaviors (extrema and intercepts), or both. During interviews, hand gestures were often used to illustrate some of these properties.

Data on the manner in which students related different notational systems of a function offer some important insights into the development of a property-oriented view. Over the course of the year, three case study participants were given 44 sets of four or five cards bearing graphs, equations, and tables of functions (see Figures 2 and 3 for examples). Some of the cards were of the same function, some were of the same function class, some shared functional properties (such as the same y-intercept or symmetry), and some were relations that were not functions. The participants were asked to 'discuss any similarities or differences' that they noticed, although they almost always focused on similarities. Generally, the participants either compared on a local, point-by-point basis or used global properties and behaviors. A local comparison strategy would involve the student checking several values of the domain on each notational system and determining if they have identical range values. This approach

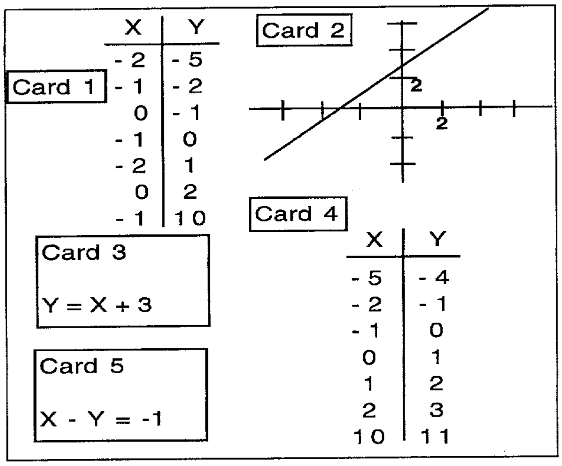

Figure 3. Translation task.

reflects an action view of function. A property-oriented strategy would discuss things such as ‘the same slope and intercepts’ or ‘both have three humps and are always positive’. This strategy would correspond to a more structural view of function. Sometimes the students would combine these strategies. For example, when relating two notational systems of a linear function, a student may notice that each has the same y-intercept (without evaluating a function or plotting a point) and then check ‘any other point’.

Overall on these tasks, the case study participants were much more inclined to use local ‘point-checking’ strategies when relating Equation-Numeric notational pairs, and more like to use global properties when relating Graphic-Equation pairs. Table 1 illustrates these tendencies in the strategies of Curt, a case study participant, on tasks before and after instructional units on linear, polynomial, and exponential functions. Because very little change occurred in the strategies across the units, it seems that the notational systems were more of a factor than the function class in determining the strategy used. The presence of a function in graphic form appeared to encourage the use of properties in this kind of analysis.

More detailed data address the issue of reification. A series of interviews with Erica illustrate how she reified the concept of linearity in terms of functional properties. Erica was confronted with a task involving the graph and equation of the functions $y = x + 8$ and $y = -2x + 8$ during the Pre-interview of the linear unit (see Figure 2):

Erica: (long pause)

DS: Which ones are you comparing now.

TABLE I
Identified translation strategies used by Curt throughout the year

Interview	Strategy	Equation-Numeric Pairs	Graphic-Numeric Pairs	Graphic-Equation Pairs
Before the Linear Unit	Local	6	4	3
	Global		2	
	Combination			
After the Linear Unit	Local	2	3	3
	Global	2	6	6
	Combination			
Before the Polynomial Unit	Local	5	2	3
	Global		2	4
	Combination		1	1
After the Polynomial Unit	Local	1		
	Global	1		5
	Combination		2	
Before the Exponential Unit	Local	3	1	1
	Global	1	3	1
	Combination	1	2	
After the Exponential Unit	Local	2	2	
	Global	1	2	
	Combination		2	2

Erica: Cards 2 and 1, they both have the y-intercept the same, and it appears that would be the solution, uh, yeah, uh (pause) Card 3 has the same intercept too, um, (pause) oh Card 3 is the solution to Card 2, um (pause) Card 1 wouldn't be but it does have the same y-intercept . . . well all of them share the same y-intercept as Card 2. Eight is the y-intercept, um (pause) and Card 1 would be the solution to Card 4.

DS: How are you comparing these?

Erica: Um, well it's kind of hard to see, but I was just putting numbers in.

DS: Ok, which numbers?

Erica: For Cards 1 and 4 I guess I put zero in for y and it would have to be four, which is what the x is on here.

DS: OK, in general, any similarities or differences?

Erica: Just that the y-intercept was the same with all of them.

DS: OK.

Although Erica is concerned with properties of the linear function, namely the intercepts, her analysis relied heavily on point-checking strategies. After the instruction in the linear unit, Erica faced a task in which Cards 2 and 3 related the graph and equation of the function $y = x + 3$, while Cards 4 and 5 were the numeric and symbolic forms of $x - y = -1$ (Figure 3):

Erica: (pause) Cards 3 and 2, um, Card 3 would be the equation of the line (Card 2).
DS: How did you know that one?
Erica: Um, because three is the y-intercept of the equation and it runs through three, and then, um, if the slope is just one the line would intercept at negative three.
DS: Ok.
Erica: (pause) Cards 5 and 4, the x and y's fit into the equation.
DS: How did you check that?
Erica: I just plugged the numbers in.

Erica is beginning to use the property of slope to make connections between the two cards, but she used slope to check the point $(-3, 0)$ and not as a defining property of the function. Functional properties were used on pairs that included a graph, whereas local point-checking strategies were used to relate the Numeric-Equation pair.

Later, on a task given before the polynomial unit (about 8 weeks after the linear unit), Erica showed signs of reifying linear functions in regard to functional properties. Cards 3 and 4 are the graph and equation of the function $y = -2x + 4$:

Erica: Card 4 is the equation for Card 3. They both have y-intercept of four and slope is -2.

Erica is satisfied that both cards describe the same function without any reliance on point-plotting or evaluation. The function is completely determined by the two properties of slope and y-intercept. Hence, a property-oriented view of function was dominate in this analysis, and the notion of linearity was constructed as a combination of specific functional properties.

The other case study students made similar strides in regard to linear and quadratic functions. However, very little evidence was found to suggest that the students reified the general notion of function as an object able to possess a variety of properties beyond those exhibited by polynomials. Eisenberg and Dreyfus (1994) ask the question, “Are process and object conceptions of function transformation acquired progressively, first for better known functions (quadratics) and later for other classes of functions?” In regard to a property-oriented view of function, the answer appears to be “Yes”.

One reason that general reification might not have taken place can be found by reexamining the nature of the instruction. Field notes and videotape analysis revealed that the types of functions which provided nearly

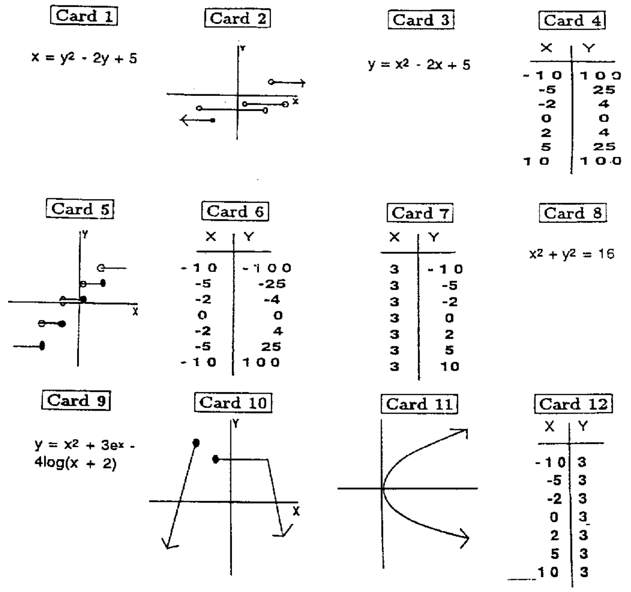

Figure 4. Examples and non-exmaples of functions.

all of the examples used in the course were the elementary polynomials common to a traditional second-year algebra course (Slavit, 1994a). This had the effect of restricting the number of properties that the students encountered, thereby restricting the ability of the students to reify function in this regard. Schwarz and Dreyfus (1995) have suggested that students who lack experience in performing actions on functions with a specific property may have difficulty incorporating these property-oriented aspects into a function concept image:

> Concept acquisition is intimately linked to actions. The lack of such actions in certain settings implies that properties may not be seen as invariant through several settings. Such properties may, for the learner, become features of formal objects (graphs, formulae or tables) rather than properties of the concept itself (p. 264).

Interview data following the polynomial unit provide further illustration of the limiting effect of the instruction. Tasks were given to the case study students that consisted of 12 cards depicting two examples and non-examples of functions in each of the numeric, graphic, and symbolic notational systems (see Figure 4). The students were asked to identify the examples and non-examples of functions. The students had little trouble on the numeric and symbolic tasks, but their understandings on the graphic examples were less clear. All three participants were misled by graphic properties and behaviors not associated with the elementary functions used in the course, particularly the finiteness of the graph or discontinuities. On Card 5 (see Figure 4) Erica stated:

Erica: It's not a function.
DS: How come?
Erica: It has to be continuous (pause).
DS: If I connected these points would that be a function?
Erica: No, you can't really get an equation that would give you that. It's more like a slope.
DS: Ok, so is the fact that it's not continuous or that you can't get an equation . . .
Erica: Both.

Erica's concept image of the graph of a function clearly involved continuous functions with infinite domains which can be expressed by a symbolic expression $y = f(x)$. We also see the manner that Erica relies on linearity, a type of function previously shown to be quite familiar to her, in a task which has very little to do with linearity. In fact, the nature of the task would appear to evoke more of a relational view than a property-oriented view in the analysis. In other words, Erica is relying on what she knows; in particular, she is relying on the functional properties that she knows.

6. DISCUSSION

This paper has provided an overview of various approaches to understanding students' conceptions of function. A property-oriented route to the reification of function was then described that extends ideas represented in many of these theories. Data were supplied to illustrate that a property-oriented perspective is just one view of function that students can develop under certain circumstances and vehicles of instruction. Other perspectives have been discussed in this paper, and collectively they provide a broad description of a concept image of function and a solid basis for descriptions of functional reasoning. Hence, these perspectives should be considered when analyzing student understandings in functional situations as well as theoretical support for curriculum development efforts.

It is clear that students initially focus on actions when developing understandings of function. An important consequence of an action view is that, by definition, it is solely concerned with local functional properties. When relationships between sets of inputs and outputs are formed, then a more object-oriented view of function can be built. But making relationships between sets of input-output ordered pairs is not an easy task (Sfard & Linchevski, 1994). The problem is compounded when functional symbolisms are confronted in very different forms (such as graphs and equations) and when functional ideas are confronted through mathematical symbols that lack a contextual basis (Filoy & Rojano, 1985). For example, when a student is asked to factor a difference of squares, what kinds of functional understandings are being built? What object-oriented understandings of function can be acquired when a student solves a linear equation of the form $ax + b = 0$? In the latter example, if the student is allowed an opportunity to reflect on the solution, what can be acquired is a relationship between the x- and y-values of the x-intercept of the function. While reflections on this relationship support the action view of function, they can do little to promote more object-oriented views of the concept unless this relationship is seen as a specific, local property that the entire function possesses. Otherwise, intercepts can become local phenomena of a function or set of functions, providing very specific information about one aspect of growth behavior or situational context. Attention should be given to the role that local function properties have in relation to the overall property-oriented view.

A look at the history of algebra reform reveals that explicit attention given to the property-oriented perspective in the development of curricula and pedagogy has been minimized (Smith, 1926; Osborne & White, 1970; Jones, 1970). The latter statement is less true for the teaching of skills and algorithms than for the development of student thinking. This situation suggests that the question of the sufficiency of the kinds of functions that are currently most often studied in today's schools be reexamined in regard to allowing students access to a broader array of functional properties. These could include discontinuities, finite domains, multivariable functions, or non-functional relations. One dimension on which this question should be addressed is the types of functions which give rise to contexts and situations that support investigation of algebraic and functional ideas. The connection of a functional property to a situational meaning can help strengthen an understanding of that property (Monk & Nemirovsky, 1994). Further, some functional properties, such as cusps and points of inflection, or not usually studied until ideas and techniques of calculus can help make them more explicit. This suggests the consideration of the future role of functions in advanced mathematics as another dimension to address the above question.

A final reason that current reform efforts should make use of this perspective is due to its visual nature. It is accepted that the teaching of geometry should begin with an intensive focus on perceptual experiences and visual analysis that lead to more abstract, formalized settings (Clements and Battista, 1992; NCTM, 1989). We have discussed how current algebra reform is relying heavily on the visual power of technologies, and the property-oriented view of function has been shown to arise in this setting. Hence, it seems that teachers and curriculum developers should be made aware of student tendencies to think and problem solve using specific functional properties. Further, the visual nature of the property-oriented perspective on functions is appealing for discussions of algebra reform at early grade levels (Kaput, 1995; in preparation). If it is accepted that the development of the function concept is central to algebraic reform and the development of algebraic thought, then investigations that allow students to relate contextual settings with numerous and varied functional properties should be part of these efforts.

NOTES

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[2] I would like to acknowledge Anna Sfard for making this point explicit to me.

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REFERENCES

Briedenbach, D. E., Dubinsky, E., Hawks, J. & Nichols, D.: 1992, ‘Development of the process conception of function’, *Educational Studies in Mathematics* 23, 247–285.

Clements, D. H. & Battista, M. T.: 1992, ‘Geometry and spatial reasoning’, in D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. The Council, Reston VA, 420–464.

Confrey, J. & Smith, E.: 1995, ‘Splitting, covariation, and their role in the development of exponential functions’, *Journal for Research in Mathematics Education* 26(1), 66–86.

Confrey, J. & Smith, E.: 1991, ‘A framework for functions: Prototypes, multiple representations, and transformations’, in R. Underhill & C. Brown (Eds.), *Proceedings of PME-NA XIII*, Blacksburg, Virginia, 57–63.

Davis, R. B.: 1975, ‘Cognitive processes involved in solving simple algebraic equations’, *Journal of Children’s mathematical Behavior* 1(3), 7–35.

Eisenberg, T. & Dreyfus, T.: 1994, ‘On understanding how students learn to visualize function transformations’, *Research on Collegiate Mathematics Education* 1, 45–68.

Filloy, E. & Rojano, T.: 1985, ‘Operating on the unknown and models of teaching’, in S. K. Damarin & M. Shelton (Eds.), *Proceedings of PME-NA VII*, Columbus, Ohio, 75–79.

Gray, D. O. & Tall, E. M.: 1994, ‘Duality, ambiguity, and flexibility: A ‘proceptual’ view of simple arithmetic’, *Journal for Research in Mathematics Education* 25(2), 116–140.

Greeno, J. G.: 1988, ‘The situated activities of learning and knowing mathematics’, in M. J. Behr, C. B. Lacampagne & M. M. Wheeler (Eds.), *Proceedings of PME-NA VIII*, Dekalb, IL, 481–521.

Jones, P. S.: 1970, ‘Present-day issues and forces’, in *A History of Mathematics Education in the United States and Canada* (32nd Yearbook of the National Council of Teachers of Mathematics). The Council, Reston, 453–465.

Judd, C.: 1928, ‘The fallacy of treating school subjects as ‘tool subjects’, in J. R. Clark & W. D. Reeve (Eds.), *Selected Topics in the Teaching of Mathematics*, Teachers College, New York, 61–79.

Kaput, J. J. (Ed.): in preparation, *Employing Children’s Natural Powers to Build Algebraic Reasoning in the Content of Elementary Mathematics*.

Kaput, J. J.: 1995, ‘A research base supporting long term algebra reform?’, in D. T. Owens, M. K. Reed & G. M. Millsaps (Eds.), *Proceedings of PME-NA XVII*, Columbus, Ohio, 71–94.

Kieren, T. E.: 1994, ‘Bonuses of understanding mathematical understanding’, in D. F. Robitaille, D. H. Wheeler & C. Kieran (Eds.), *Proceedings of ICME-VII*, Sainte-Foy, 211–228.

Kieren, T. E.: 1990, ‘Understanding for teaching for understanding’, *The Alberta Journal of Educational Research* 36(3), 191–201.

Kieren, T. E. & Pirie, S. E. B.: 1991, ‘Recursion and the mathematical experience’, in L. Steffe (Ed.), *The Epistemology of Mathematical Experience*. New York, Springer-Verlag.

Magidson, S.: 1992, From the laboratory to the classroom: A technology-intensive curriculum for functions and graphs’, *Journal of Mathematical Behavior* 11, 361–376.

Monk, S. & Nemirovsky, R.: 1994, ‘The case of Dan: Student construction of a functional situation through visual attributes’, *Research on Collegiate Mathematics Education* 1, 139–168.

Nemirovsky, R. & Rubin, A.: 1992, ‘Students’ tendency to assume resemblances between a function and its derivative’, TERC Working Paper 2–92, Cambridge, Massachusetts.

National Council of Teachers of Mathematics (NCTM): 1989, *Curriculum and Evaluation Standards for School Mathematics*. The Council, Reston, VA.

Osborne, A. R. & Crosswhite, F. J.: 1970, ‘Mathematics education on the defensive: 1920–1945’, in *A History of Mathematics Education in the United States and Canada* (32nd Yearbook of the National Council of Teachers of Mathematics). The Council, Reston, 197–234.

Piaget, J., Grize, J., Szeminska, A. & Bang, V.: 1977, *Epistemology and Psychology of Functions*, Reidel, Boston.

Quesada, A. R. & Maxwell, M. E.: 1994, ‘The effects of graphing calculators to enhance college students’ performance in precalculus’, *Educational Studies in Mathematics* 27, 205–215.

Rich, B.: 1990, ‘The effect of the use of graphing calculators on the learning of function concepts in precalculus mathematics’, *Dissertation Abstracts International* 52, 835A.

Ruthven, K.: 1990, ‘The influence of graphic calculator use on translation from graphic to symbolic forms’, *Educational Studies in Mathematics* 21(5), 431–450.

Schoenfeld, A. H., Smith, J. P. & Arcavi, A.: 1993, ‘Learning: The microgenetic analysis of one student’s evolving understanding of a complex subject matter domain’, in R. Glaser (Ed.), *Advances in Instructional Psychology*, Vol. 4, Erlbaum, Hillsdale, NJ.

Schwarz, B. & Dreyfus, T.: 1995, ‘New actions upon old objects: A new ontological perspective on functions’, *Educational Studies in Mathematics* 29, 259–291.

Schwarz, B., Dreyfus, T. & Bruckheimer, M.: 1990, ‘A model of the function concept in a three-fold representation’, *Computers and Education* 14(3), 249–262.

Sfard, A.: 1992, ‘Operational origins of mathematical objects and the quandary of reification: The case of function’, in G. Harel and E. Dubinsky (Eds.), *The Concept of Function:*

Aspects of Epistemology and Pedagogy. Mathematical Association of America, Notes Number 25, 59–84.

Sfard, A.: 1991, 'On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin', *Educational Studies in Mathematics* 22, 1–36.

Sfard, A.: 1989, 'Transition from operational to structural conception: The notion of function revisited', in G. Vergnaud, J. Rogalski & M. Artigue (Eds.), *Proceedings of PME-XIII*, Vol. III, Laboratoire PSYDEE, Paris, 151–158.

Sfard, A. & Linchevski, L.: 1994, 'The gains and pitfalls of reification – The case of algebra', *Educational Studies in Mathematics* 26, 191–228.

Slavit, D. B.: 1996, 'Graphing calculators in a 'hybrid' Algebra II classroom', *For the Learning of Mathematics* 16(1), 9–14.

Slavit, D. B.: 1995, 'A growth-oriented route to the reification of function', in *Proceedings of PME-NA XVII*, Columbus, Ohio, 284–290.

Slavit, D. B.: 1994a, 'Algebra Instruction Using the Graphing Calculator and its Effect on Students' Conceptions of Function', *Dissertation Abstracts International*.

Slavit, D. B.: 1994b, 'The Effect of Graphing Calculators on Students' Conceptions of Function', paper presented at the annual meeting of the American Educational Research Association, New Orleans.

Smith, D. E.: 1926, 'A general survey of the progress of mathematics in our high schools in the last twenty-five years', in *A General Survey of Progress in the Last Twenty-Five Years* (1st Yearbook of the National Council of Teachers of Mathematics). The Council, Reston, 1–31.

Teles, E. J.: 1989, 'Numerical and graphical presentation of functions in precalculus', *Dissertation Abstracts International* 51, 777A.

Thompson, P. W.: 1994, 'Students, functions, and the undergraduate curriculum', in E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in Collegiate Mathematics Education I*, Providence, R.I., American Mathematical Society, and Washington, D.C., Mathematical Association of America.

Thompson, P. W. & Sfard, A.: 1994, 'Problems of reification: Representations and Mathematical objects', in D. Kirshner (Ed.), *Proceedings of PME-NA XVI*, Baton Rouge, 1–32.

Tierney, C. C. & Nemirovsky, R.: 1995, 'Children's graphing of changing situations', paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Vinner, S.: 1983, 'Concept definition, concept image and the notion of function', *International Journal of Mathematics Education in Science and Technology* 14(3), 293–305.

Mathematics Education,
Washington State University,
322 Neill Hall,
Pullman, WA 99164–3113,
U.S.A.