A family of covering properties for forcing axioms and supercompact cardinals

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PROBLEM

MM and PFA appears to produce models of set theory in which every "*consistent*" set of size at most $\aleph_1$ "*exists*".

How to formulate this in a suitable form?

For example in this way:

Theorem 1 (Veličković) *Assume* MM. *Let* W *be an inner model such that* $\omega_2^W = \omega_2$*. Then* $P(\omega_1) \subseteq W$*.*

Theorem 2 (Caicedo, Vel.) *Assume* $W \subseteq V$ *are models of* BPFA *such that* $\omega_2^W = \omega_2$*. Then* $P(\omega_1) \subseteq W$*.*

We want to extend these results all over the cardinals:

Conjecture 1 (Caicedo, Veličković) *Assume* $W \subseteq V$ *are models of* MM *with the same cardinals. Then* $[Ord]^{\leq\omega_1} \subseteq W$*.*

This is almost best possible, since:

- There exist $W \subseteq V$ models of MM with the same cardinals such that $[Ord]^{\omega_2} \not\subseteq W$.

- Using stationary tower forcing it is possible to produce two models of MM, $W \subseteq V$ such that $[Ord]^{\leq\omega_1} \not\subseteq W$. However the two models have different cardinals.

FIRST PROBLEM TO MATCH: FIXING THE COFINALITIES.

If g is a generic Prikry sequence on a measurable κ, $V \subseteq V[g]$ have the same cardinals but $g \notin V$ is a countable set of ordinals.

This first counterexample is overruled by the next two theorems:

Theorem 3 (Cummings, Schimmerling)

Assume $W \subseteq V$ *are models of* ZFC *and:*

- κ *is regular in* W
- $\mathrm{cf}(\kappa) = \omega$
- $(\kappa^+)^W = \kappa^+$

Then $\Box_{\kappa,\omega}$ *holds.*

Theorem 4 (Todorčević, Magidor) PFA *implies that* $\Box_{\kappa,\omega}$ *fails.*

We must deal now with the case that the first κ which is regular in W and singular in V has cofinality ω_1. This can happen:

Theorem 5 (Gitik) *There are $W \subseteq V$ models of* ZFC *with the same cardinals, the same bounded subsets of κ and such that κ is regular in W and has cofinality ω_1 in V.*

Cummings and Schimmerling result do not cover this case.

The techniques to treat this case are common to models of a forcing axiom or of a strongly compact cardinal.

Theorem 6 (V.) *Assume* MM. *Let* κ *be singular (and strong limit). Let* W *be an inner model such that* κ *is regular in* W *and* $\kappa^+ = (\kappa^+)^W$. *Then* $\mathrm{cf}(\kappa) > \omega_1$.

Theorem 7 (V.) *Assume* λ *is strongly compact and* $\kappa > \lambda$. *Let* W *be an inner model such that* κ *is regular in* W *and* $\kappa^+ = (\kappa^+)^W$.
Then $\mathrm{cf}(\kappa) \geq \lambda$.

These results settle the problem with cofinalities.

Corollary 8 *Let V be a model of* MM *(such that every limit cardinal is strong limit). Let W be an inner model with the same cardinals. If κ is regular in W,* $\mathrm{cf}(\kappa) > \omega_1$.

Corollary 9 *Assume λ is strongly compact. Then any forcing which preserves the relevant cardinals and changes the cofinality of some regular $\kappa > \lambda$ to some $\delta < \lambda$ will destroy the strong compactness of λ.*

Since all known means to change cofinalities presume that some Prikry sequence is appearing somewhere, these results raise the following question:

Problem 1 *Assume $W \subseteq W$ are models of* MM *with the same cardinals. Do they have the same cofinalities?*

Gitik's theorem shows that MM or an axiom of this strength is needed. A negative answer needs a completely new idea for changing the cofinality of a regular cardinal.

Back to the conjecture:

Corollary 10 *Assume $W \subseteq V$ have the same cardinals, V models* MM *and is a set-forcing extension of W. Then $[Ord]^{\leq \omega_1} \subseteq W$.*

This is currently my best answer to the conjecture.

COVERING MATRICES

For a regular cardinal λ and a successor $\kappa^+ > \lambda$, $\mathcal{D} = (D(\alpha,\xi) : \alpha < \lambda, \xi \in \kappa^+)$ is a λ-covering matrix for κ^+ if:

(i) $\xi = \bigcup_{\alpha<\lambda} D(\alpha,\xi)$ for all $\xi < \kappa^+$,

(ii) $|D(\alpha,\xi)| < \kappa$ for all $\xi < \kappa^+$ and $\alpha < \lambda$,

(iii) $D(\alpha,\xi)$ is strictly contained in $D(\beta,\xi)$ for all $\xi < \kappa^+$ and for all $\alpha < \beta < \lambda$,

(iv) for all $\eta < \xi < \kappa^+$ and for all $\alpha < \lambda$, there is $\beta < \lambda$ such that $D(\alpha,\eta) \subseteq D(\beta,\xi)$,

(v) for all $X \in [\kappa^+]^{\leq\lambda}$, there is $\gamma_X < \kappa^+$ such that for all $\eta < \kappa^+$ and $\alpha < \lambda$, there is $\beta < \lambda$ such that $D(\alpha, \eta) \cap X \subseteq D(\beta, \gamma_X)$.

(v-S) $\mathcal{D}$ is strongly λ-covering if for all $\eta < \xi < \kappa^+$ and $\alpha < \lambda$, there is $\beta < \lambda$ such that $D(\alpha, \xi) \cap \eta \subseteq D(\beta, \eta)$.

EXAMPLES OF COVERING MATRICES

The matrices provided by the lemma below are useful to prove SCH from PFA.

Lemma 11 *Assume that* $\kappa > 2^\lambda$ *is a singular cardinal of cofinality* λ. *Then there is a* λ-*covering matrix* $\mathcal{C}$ *for* κ^+ *with* $\beta_\mathcal{C} = \kappa$.

The following lemma is a key-step in the proof of all the theorems concerning a change of cofinalities:

Lemma 12 *(Cumm., Schimm.-Tod.-Galvin...) Assume that* κ *is a regular cardinal. Then there is a strongly* κ*-covering matrix* $\mathcal{C}$ *for* κ^+ *with* $\beta_{\mathcal{C}} = \kappa$.

Remark 13 *Assume* κ *is singular of cofinality* λ*, but that* κ *is regular in an inner model* W *such that* $\kappa^+ = (\kappa^+)^W$*. Then a strongly* κ*-covering matrix* $\mathcal{D} \in W$ *on* κ^+ *can be refined to a strongly* λ*-covering matrix* $\mathcal{E}$ *on* κ^+.

The last lemma below is useful to obtain a proof that $\square_\kappa$ fails under forcing axioms or suitable large cardinal assumptions:

Lemma 14 *(Todorčević) Assume* $\square_\kappa$. *Then there is a strongly* ω*-covering matrix* $\mathcal{C}$ *for* κ^+ *with* $\beta_{\mathcal{C}} = \kappa$.

A FAMILY OF COVERING PROPERTIES

Definition 15 CP(λ, κ) *holds if for every* λ*-covering matrix* $\mathcal{D} = \{D(\alpha, \eta) : \alpha < \lambda, \eta < \kappa\}$ *on* κ^+*, there is* A *unbounded subset of* κ^+ *such that* $[A]^\lambda$ *is covered by* $\mathcal{D}$*.*

$[A]^\lambda$ is covered by $\mathcal{D}$ means that every $X \in [A]^\lambda$ is contained in some $D(\alpha, \eta)$.

Fact 1 CP(λ, κ) *implies that there is no strongly λ-covering matrix $\mathcal{D}$ on κ^+.*

Proof: Assume to the contrary that A is an unbounded subset of κ^+ and $[A]^\lambda$ is covered by

$$\mathcal{D} = \{D(\alpha, \eta) : \alpha < \lambda, \eta < \kappa^+\}.$$

Pick an initial segment of A of type κ and let $\eta < \kappa^+$ be its supremum.

Now $A \cap \eta \not\subseteq D(\alpha, \eta)$ for all $\alpha < \lambda$ since

$$\text{otp}(A \cap \eta) = \kappa > |D(\alpha, \eta)|$$

for all $\alpha < \lambda$ by property *(ii)* of $\mathcal{D}$.

Thus we can produce X subset of $A \cap \eta$ of size λ such that $X \setminus D(\alpha, \eta)$ is non-empty for all $\alpha < \lambda$.

Since $[A]^\lambda$ is covered by $\mathcal{D}$ we can find $\nu < \kappa^+$ and $\alpha < \lambda$ such that $X \subseteq D(\alpha, \nu)$.

Now by property *(v-S)* we have that for some $\beta < \lambda$, $D(\alpha, \nu) \cap \eta \subseteq D(\beta, \eta)$.

Thus $X \subseteq D(\beta, \eta)$. Contradiction. $\square$

Fact 2 CP(ω, κ) *implies* $\square_\kappa$ *fails. Moreover if* CP(ω, κ) *holds for all singular* κ *of countable cofinality, then the singular cardinal hypothesis is true.*

Fact 3 CP(κ, κ^+) *fails.*

Theorem 16 (V.) *If* λ *is strongly compact* $\mathrm{CP}(\theta, \kappa)$ *holds for all regular* $\theta < \lambda$ *and* $\kappa \geq \lambda$*.*

Theorem 17 (V.) PFA *implies* $\mathrm{CP}(\omega, \kappa)$ *for all uncountable cardinals* κ*.*

MM implies CP(ω_1, κ) for all uncountable κ?

I don't know. Nonetheless MM rules out the possibility that a regular cardinal can change its cofinality to $\aleph_1$ while the successor is preserved.

This can be proved using an ad hoc elaboration on Cummings and Schimmerling result.

What is interesting of the proof is that it uses a large fragment of MM to build a $\square_{\kappa,\aleph_1}$-square sequence.

The strong Chang conjecture SCC(λ) holds if for every club $\mathcal{D}$ of countable structures M there is $\mathcal{C}$ club subset of $\mathcal{D}$ such that for every $M \in \mathcal{C}$ and $\alpha < \lambda$, there is $N \in \mathcal{C}$ such that $M \cap \omega_1 = N \cap \omega_1$ and $\alpha < \sup(N \cap \lambda)$.

Theorem 18 (V.) *Assume:*

- κ *is singular of cofinality* $\aleph_1$,
- SCC(κ^+),
- κ *is regular in an inner model* W *such that* $\kappa^+ = (\kappa^+)^W$.

Then $\square_{\kappa,\aleph_1}$ *holds.*