A brief introduction to Ω-logic

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Truth in set theory

First of all I will adopt for this talk a *platonistic* attitude.

> *There is one standard model of set theory:*
>
> *Our axiomatizations of set theory in the first order language* $\{\in, =\}$ *are a useful mean to describe its properties.*

GOAL: isolate a class of structures which are as close as possible to the *standard* model of set theory and study validity of formulas with respect to this class.

Given Γ family of structures for the language of set theory, we define the notions of Γ-validity, Γ-consistency and $\models_\Gamma$ of Γ-logical consequence as usual, i.e.:

- ϕ is Γ-valid if it holds in all structures in Γ,
- a theory S is Γ-consistent if it holds in some structure in Γ,
- $T \models_\Gamma S$ if all structures in Γ which are models of T are also models of S.

Definition

Let T be a theory extending ZFC, S be a set of logical sentences, Γ a family of structures for the language of set theory. T is a Γ-solution for S if for all $\phi \in S$, $T \models_\Gamma \phi$ or $T \models_\Gamma \neg\phi$.

Goal: isolate a class Γ of structures which are as close as possible to the *standard* model of set theory and theories T such that:

- T is Γ-consistent,
- T is a Γ-solution for the largest possible family S of logical statements.

First interesting class: WF, the family of transitive models N of ZFC such that $\omega_1 + 1 \subseteq N$

.

These models are correctly representing the $\in$-relation since in these models this relation is extensional and well founded as prescribed by the axioms of extensionality and foundation.

If there are models of ZFC there are certainly ill-founded (i.e. non-transitive) models of ZFC by the compactness theorem.

If there are transitive models of ZFC there are certainly countable extensional and well founded models of ZFC by the downward Löwenheim-Skolem theorem.

WF is a proper subclass of the first order models of ZFC and WF-validities are a proper superset of the first order theorems of ZFC.

First important absoluteness result.

Theorem (Shoenfield absoluteness)

Assume a Σ^1_2-statement holds in a transitive models of ZFC *containing* $\omega_1 + 1$. *Then it holds in all transitive models of* ZFC *containing* $\omega_1 + 1$, *i.e.: $\emptyset$ is a* WF*-solution for the family of Σ^1_2-sentences.*

Riemann's hypothesis is a Σ^1_2-statement......
To what extent can we establish absoluteness for more complex statements?

WF is an hardly tractable class for it is not possible to define $\models_{WF}$ in a first order way inside a transitive model V:

> *There are potentially too many transitive models of* ZFC *in order that one transitive model is able to describe somehow all the other transitive model.*

Moreover it is known that Shoenfield absoluteness lemma is best possible at least in ZFC:

Theorem (Folklore?)

There are Σ^1_3-statements which can have different truth-values in different transitive models of ZFC.

Boolean valued models

Let V be a transitive model of ZFC and $\mathbb{B} \in V$ be a complete boolean algebra in V.
Define the boolean valued model $V^{\mathbb{B}}$ by transfinite recursion as follows:

- $V^{\mathbb{B}}_{\emptyset} = \emptyset$
- $V^{\mathbb{B}}_{\alpha+1} = \{f : f \texttt{ is a partial function from } V^{\mathbb{B}}_{\alpha} \texttt{ into } \mathbb{B}\}$
- $V^{\mathbb{B}}_{\beta} = \bigcup_{\alpha<\beta} V^{\mathbb{B}}_{\alpha}$ if β is limit.

- If $\mathbb{B} = 2$, $V^{\mathbb{B}}$ is the set of characteristic functions of sets in V and thus is an isomorphic copy of V.
- Intuitively for $f \in V^{\mathbb{B}}_{\alpha+1}$ and $g \in dom(f)$, $f(g) \in \mathbb{B}$ is the boolean value $\|g \in f\|_{\mathbb{B}}$ the model $V^{\mathbb{B}}$ assigns to the formula $g \in f$.

We shall work in the expanded language $\{\in, =\} \cup V^{\mathbb{B}}$ and we assign a boolean value to any formula in this language. What is difficult is to assign the correct boolean value to the atomic formulas $f = g$ and $f \in g$ as f, g varies over $V^{\mathbb{B}}$.

Any formula can be evaluated using the completeness of $\mathbb{B}$.
For example:

- $\|\phi \to \psi\|_{\mathbb{B}} = \|\neg\phi \vee \psi\|_{\mathbb{B}} = \|\phi\|_{\mathbb{B}}^{c} \cup \|\psi\|_{\mathbb{B}}$,
- $\|\forall x \psi(x, \vec{f})\|_{\mathbb{B}} = \inf\{\|\psi(g, \vec{f})\|_{\mathbb{B}} : g \in V^{\mathbb{B}}\}$ (we use completeness of $\mathbb{B}$ here)
- $\|\exists x \psi(x, \vec{f})\|_{\mathbb{B}} = \sup\{\|\psi(g, \vec{f})\|_{\mathbb{B}} : g \in V^{\mathbb{B}}\}$
- ......

Clearly boolean valued models are sound for first order calculus.

Main forcing theorem:

Theorem (Cohen, Scott, Solovay)

Assume V is a transitive model of ZFC *and $\mathbb{B} \in V$ is a complete boolean algebra. Then $\|\phi\|_{\mathbb{B}} = 1_{\mathbb{B}}$ for all axioms ϕ of* ZFC.

The boolean value of an arbitrary statement ϕ heavily depends on the combinatorial properties of $\mathbb{B}$. The family of structures $(\Omega)^{V}$ depends to a great extent on the choice of V.

All mathematically interesting independence results in set theory are obtained using set forcing i.e. boolean valued models!!!!!!

Ω-LOGIC

Definition

Given a transitive model V of ZFC, ϕ is Ω-valid in V $(\models_\Omega \phi)^V$ iff $\|\phi\|_{\mathbb{B}} = 1_{\mathbb{B}}$ for all boolean algebras $\mathbb{B} \in V$ i.e. ϕ holds in all boolean valued models $V^{\mathbb{B}}$.

$(\models_\Omega \phi)^V$ *is first order expressible in* V*, i.e it holds if and only if:*

$$V \models \forall \mathbb{B} \|\phi\|_{\mathbb{B}} = 1_{\mathbb{B}}.$$

For a given transitive structure V of ZFC, $(\models_\Omega)^V$ is defined using the meta-class $(\Omega)^V$ defined in the meta-theory as the family of all classes which are boolean valued models definable in V, however we can argue the following:

> *A transitive model* V *of* ZFC *is able to describe all boolean valued model of the class* $(\Omega)^V$*.*

This is in sharp contrast with the situation for the meta class WF.

To simplify notation I shall omit any relativization of $(\models_\Omega \phi)^V$ and write instead $\models_\Omega \phi$ since:

> *All theorems about $\models_\Omega \phi$ are established just on the basis of large cardinal axioms.*
> *Any transitive model V which has enough large cardinals will satisfy these properties about his notion of $(\models_\Omega \phi)^V$.*

I will not go into details but the core of Cohen's forcing theorem brings as an outcome that the class $(\Omega)^V$ of boolean valued models $V^{\mathbb{B}}$ (via passing to the quotient structures given by V-generic filters for $\mathbb{B}$) is a subclass of WF.
Thus for any transitive model V of ZFC, $(\models_\Omega)^V$ is a stronger notion than $\models_{\mathrm{WF}}$.
It is questionable though whether we can drop out all models which are not in $(\Omega)^V$ in our attempt to analyze truth in set theory.
This is one of the objections to Ω-logic.

SOME OF THE MAIN RESULTS OF WOODIN

Let ZFC^* be ZFC enriched with all large cardinal axioms we might like: for example ZFC+*there are class many supercomapct cardinals*.
This theory is much stronger than what is needed for all the results I shall list below.
Solidity of the notion of Ω-validity:

Theorem

Assume $V \models \mathrm{ZFC}^*$ *and is transitive. Then for every statement* ϕ *the following are equivalent:*

- $(\models_\Omega \phi)^V$,
- *there is some complete boolean algebra* $\mathbb{B} \in V$ *such that* $(\models_\Omega \phi)^{V^\mathbb{B}}$ *i.e.* $\|(\forall \mathbb{Q} \|\phi\|_\mathbb{Q} = 1_\mathbb{Q})\|_\mathbb{B} = 1_\mathbb{B}$,
- *for all complete boolean algebra* $\mathbb{B} \in V$, $(\models_\Omega \phi)^{V^\mathbb{B}}$.

The philosophical appeal of Ω-logic would have been very poor if $(\models_\Omega)^V \neq (\models \Omega)^{V^{\mathbb{B}}}$ for some boolean algebra $\mathbb{B}$.
The point in Ω-logic is that a priori all structures in Ω are equally fine to approximate the *true* universe of set theory.
We do not want to have a priori any reason to choose which among V and $V^{\mathbb{B}}$ resembles more the standard model of set theory.
The theorem says that even if $(\Omega)^V$ and $(\Omega)^{V^{\mathbb{B}}}$ are different families of structures, $(\models_\Omega)^V = (\models_\Omega)^{V^{\mathbb{B}}}$ provided in V there are enough large cardinals.

Ω-truth works effectively

Theorem

Assume ZFC holds in V. Then $Th(L(\mathbb{R}^V)) = Th(L(\mathbb{R}^{V^{\mathbb{B}}}))$ for all complete boolean algebras $\mathbb{B} \in V$. In particular $\emptyset$ is an Ω-solution for all statements relativized to $L(\mathbb{R})$.*
Moreover $ZFC^* \vdash AD^{L(\mathbb{R})}$.

AD settles essentially any concrete problem regarding projective sets.
Ω-logic says *AD* is the correct axiom for $L(\mathbb{R})$!!!!!.

Theorem

Assume ZFC* *holds in V. Assume ϕ is a property which is provably Δ^2_1 in* ZFC*. *Then either $\models_\Omega \phi$ or $\models_\Omega \neg\phi$. In particular $\emptyset$ is an Ω-solution for Δ^2_1-statements.*

Notice that Δ^2_1-properties correspond to the Δ_1-properties of the structure $(V_{\omega+2}, \in)$ while Σ^2_1-properties correspond to Σ_1-properties of the structure $V_{\omega+2}$.

The above theorem is best possible since CH is a Σ^2_1-statement and:

Theorem (Levy, Solovay)

For any given large cardinal axiom holding in V there are complete boolean algebras $\mathbb{B}_0, \mathbb{B}_1 \in V$ such that the large cardinal axiom holds in $V^{\mathbb{B}_i}$ and $V^{\mathbb{B}_0}$ models CH *while $V^{\mathbb{B}_1}$ does not.*

In ZFC*, $\emptyset$ cannot be an Ω-solution for CH or more generally for Σ^2_1-sentences.
However there are statements ϕ such that, in ZFC*, ϕ is an Ω-solution for a class of sentences which includes CH.

There are two most popular way to stratify the universe of sets:

- the Von Neumann hierarchy given by the iteration of the power set operation which generates the strucures V_α,
- the hierarchy of the sets H_κ given by closure with respect to more and more instances of full replacement (recall that H_κ is the set of all sets whose transitive closure have size less than κ).

Theorem

Assume ZFC* *holds in* V. *Assume* ϕ *is a* Σ^2_1*-property. If* ϕ *is* Ω*-consistent, then* CH $\models_\Omega \phi$.

This means that CH decides in Ω-logic the Σ_1-theory of $V_{\omega+2}$.

Theorem

Assume ZFC* *holds in* V. *There is an axiom* $(*)$ *which decides in* Ω*-logic the theory of* $H_{\aleph_2}$, *i.e for every sentence* ϕ *either* $(*) \models_\Omega (H_{\aleph_2} \models \phi)$ *or* $(*) \models_\Omega (H_{\aleph_2} \models \neg\phi)$.
Moreover ZFC$^* + (*) \vdash 2^{\aleph_0} = \aleph_2$.

CONCLUSION.....

If we use Von Neumann stratification of the universe Ω-logic leads us to accept CH as true.
On the other hand if we use the H_κ hierarchy Ω-logic suggests that CH is false........
There is still much to speculate on the true meaning of Ω-logic and whether it will be able to settle the continuum problem.
The non-platonists can be happy of this, but the platonist will certainly be satisfied of all the absoluteness that Ω-logic provides for problems which are strictly simpler than CH.

Thanks for your attention.