Reflection and approachability

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A sample of our results:

Theorem 1. *Assume:*

- $2^{\aleph_{n-1}} < \aleph_\omega$,
- *Every countable family of* $S^{\aleph_n}_\omega$ *reflects jointly on some* $\delta < \aleph_n$.

Then club many points in $\aleph_{\omega+1}$ *of cofinality* $\aleph_n$ *are approachable.*

The reflection hypothesis are satisfied for all $n > 1$ under MM.

This is a partial answer to a problem asked in:

M. Foreman and M. Magidor, A very weak square principle, JSL 1997(1), 175-196.

Is it consistent to have a stationary set of non-approachable points of cofinality $\aleph_2$ in $\aleph_{\omega+1}$?

Theorem 2. *Assume* $\aleph_\omega$ *is strong limit and* PFA *holds. Then club many points in* $\aleph_{\omega+1}$ *of cofinality* $\aleph_n$ *are approachable for all* $n > 2$.

Theorem 3. *Assume:*

- λ *is weakly compact,*
- $\kappa > \lambda$ *is a singular cardinal of cofinality* $\theta < \lambda$.

Then club many points in κ^+ *of cofinality* λ *are approachable.*

Compare these results with the following:

Theorem 4 (Shelah)**.** *Assume:*

- λ *is strongly compact,*
- $\kappa > \lambda$ *is a singular cardinal of cofinality* $\theta < \lambda$*.*

Then there are stationarily many cofinalities γ *below* λ *such that stationarily many points in* $S^{\kappa^+}_\gamma$ *are not approachable.*

Theorem 5 (Magidor)**.** *If* MM *holds, there are stationarily many non-approachable points in* $\aleph_{\omega+1}$ *of cofinality* $\aleph_1$*.*

Chang conjectures for singular cardinals

Cummings asked in:

J. Cummings, Collapsing successors of singulars, PAMS, 125(9), 1997, 2703-2709

Is it consistent that $(\kappa^+, \kappa) \twoheadrightarrow (\aleph_2, \aleph_1)$ *for a singular* κ *of countable cofinality?*

Another application of our results is the following:

Theorem 6. *Assume* MM. *Then*

$$(\kappa^+, \kappa) \twoheadrightarrow (\aleph_2, \aleph_1)$$

fails for all singular κ *of cofinality at most* $\aleph_1$*.*

The approachability ideal $\mathcal{I}[\lambda]$ has been introduced by Shelah in his analysis of singular cardinals combinatorics.

Three results for $\mathcal{I}[\lambda]$ when λ is the successor of a singular κ:

- There is a stationary set in $\mathcal{I}[\lambda]$. This has been used to prove the existence of scales.

- $\mathcal{I}[\lambda] = P(\lambda)$ unless very large cardinals are behind the scene.

- Preservation of stationarity of S under λ-closed forcing: requires slightly more than $S \in \mathcal{I}[\lambda]$.

Some definitions

Let $\mathcal{A} = \{a_\alpha : \alpha < \kappa^+\} \subseteq [\kappa^+]^{<\kappa}$ be given,

- δ is weakly approachable with respect to $\mathcal{A}$ if there is H unbounded in δ of minimal order type such that $\{H \cap \gamma : \gamma < \delta\}$ is covered by $\{a_\alpha : \alpha < \delta\}$,

- δ is approachable with respect to $\mathcal{A}$ if there is H unbounded in δ of minimal order type such that $\{H \cap \gamma : \gamma < \delta\} \subseteq \{a_\alpha : \alpha < \delta\}$.

Definition 7. *Let κ be a singular cardinal.*

- *S is (weakly) approachable if there is a sequence*

$$\mathcal{A} = \{a_\alpha : \alpha < \kappa^+\} \subseteq [\kappa^+]^{<\kappa}$$

and a club C such that δ is (weakly) approachable with respect to α for all $\delta \in S \cap C$.

- *$\mathcal{I}[\kappa^+]$ is the ideal generated by approachable sets,*

- *$\mathcal{I}[\kappa^+, \kappa]$ is the ideal generated by weakly approachable sets.*

The following holds:

- $\mathcal{I}[\kappa^+] \subseteq \mathcal{I}[\kappa^+, \kappa]$,
- $\mathcal{I}[\kappa^+] = \mathcal{I}[\kappa^+, \kappa]$ if κ is strong limit,
- $\mathcal{I}[\kappa^+, \kappa]$ serves the same purposes of $\mathcal{I}[\kappa]$ in many of the applications of these ideals, for example the first two of the three listed before.

We will thus study $\mathcal{I}[\kappa^+, \kappa]$.

An idea of the proofs:

We sketch a proof of the following:

Theorem 8. *Assume* MM. *Then*

$$S^{\aleph_{\omega+1}}_{\aleph_n} \in \mathcal{I}[\aleph_{\omega+1}, \aleph_\omega]$$

for all $n > 2$.

This will give an idea of the general argument.

A different characterizaton of the approachability ideal

Given

$$d : [\aleph_{\omega+1}]^2 \to \omega$$

- d is **normal** if

  $$D(i, \beta) = \{\alpha < \beta : d(\alpha, \beta) \leq i\}$$

  has size less than $\aleph_\omega$ for all i and β,

- d is **transitive** if whenever $\alpha \in D(i, \beta)$

  $$D(i, \alpha) \subseteq D(i, \beta)$$

  for all $\alpha \leq \beta$ and i,

- δ of uncountable cofinality λ is d-**approachable** if there is H unbounded in δ such that $[H]^{<\lambda}$ is covered by the family:

  $$\{D(i, \beta) : i < \omega, \beta < \delta\}.$$

The following is an equivalent definition of the ideal $\mathcal{I}[\aleph_{\omega+1}, \aleph_\omega]$:

Property 9. *(Shelah) TFAE:*

- $S \in \mathcal{I}[\aleph_{\omega+1}, \aleph_\omega]$,
- *there is a normal and transitive d, and a club C in $\aleph_{\omega+1}$ such that δ is d-approachable for all $\delta \in S \cap C$.*

This characterization of the ideal is not specific for $\aleph_\omega$ and works for all singular cardinals of any cofinality.

We are led to analyze colorings

$$d : [\aleph_{\omega+1}]^2 \to \omega$$

or equivalently the matrices

$$\mathcal{D}(d) = \{D(i, \beta) : i < \omega, \beta < \aleph_{\omega+1}\}$$

where

$$D(i, \beta) = \{\alpha < \beta : d(\alpha, \beta) \leq i\}$$

We have now a strategy to show that a $\delta < \aleph_{\omega+1}$ of uncountable cofinality is d-approachable:

1. Pick A cofinal subset of δ of minimal order type,

2. take $\mathcal{E}$ to be the transitive collapse of the structure $\{D(i, \alpha) \cap A : i < \omega, \alpha \in A\}$,

3. find G unbounded subset of $\aleph_2$ such that all its initial segments are covered by $\mathcal{E}$,

4. pull back G through the inverse of the transitive collapse of A to an unbounded subset L of A which is covered by
$$\{D(i, \alpha) : i < \omega, \alpha < \delta\},$$
and use the previous fact to argue that δ is d-approachable.

the hard work is now only in part 3.

Covering matrices

$\mathcal{D}(d)$ is an example of an ω-covering matrix on $\aleph_{\omega+1}$ and $\mathcal{E}$ an example of an ω-covering matrix on $\aleph_2$.

Definition 10. ,

$$\mathcal{D} = \{D(i, \beta) : i < \omega,\ \beta \in \lambda\}$$

is an ω-covering matrix for λ if:

(i) $\beta = \bigcup_{i<\omega} D(i, \beta)$ *for all* $\beta < \lambda$,

(ii) $D(i, \beta) \subseteq D(j, \beta)$ *for all* $\beta < \lambda$ *and for all* $i < j < \omega$,

(iii) for all $\beta < \gamma < \lambda$ *and for all* $i < \omega$, *there is* $j < \omega$ *such that* $D(i, \beta) \subseteq D(j, \gamma)$.

- $\mathcal{D}$ is **transitive** if $\alpha \in D(i,\beta)$ implies $D(i,\alpha) \subseteq D(i,\beta)$.

- $\mathcal{D}$ is **closed** if for all $i < \omega$, $\beta < \lambda$ and $X \in [D(i,\beta)]^\omega$, $\overline{X} \subseteq D(i,\beta)$,

- $\beta_{\mathcal{D}} \leq \lambda$ is the least β such that for all i and γ, otp$(D(i,\gamma)) < \beta$,

- $\mathcal{D}$ is normal if $\beta_{\mathcal{D}} < \lambda$.

If d is normal and transitive $\mathcal{D}(d)$ is an example of a transitive ω-covering matrix $\mathcal{D}$ on $\aleph_{\omega+1}$ with $\beta_{\mathcal{D}} = \kappa$.

$\mathcal{E}$ is an example of a transitive θ-covering matrix on cf(δ) (a priori we can't say much on the value of $\beta_{\mathcal{E}}$).

In this talk we are interested in closed, transitive ω-covering matrices.

Lemma 11. *There is a closed, normal and transitive ω-covering matrix $\mathcal{D}$ on $\aleph_{\omega+1}$ with $\beta_{\mathcal{D}} = \aleph_\omega$.*

Remark 12. *Consider the previous example of the matrix* $\mathcal{E}$ *obtained by the transitive collapse of*

$$\{D(i,\alpha) \cap A : i < \omega, \alpha \in A\}$$

where A *is a subset of* δ *of minimal order type.*

Provided that A *is a club in* δ*, the matrix* $\mathcal{E}$ *inherits the property of being a closed, transitive* ω*-covering matrix on* cf(δ)*.*

What we need to prove is the following:

For every closed and transitive ω-covering matrix $\mathcal{E}$ on cf(δ), *there is an unbounded subset of* cf(δ) *such that all its initial segments are covered by $\mathcal{E}$.*

This will be the case provided that cf(δ) satisfies some mild reflection principles.

Definition 13. *Let* $\mathcal{D} = \{D(i, \alpha) : i < \omega, \alpha < \lambda\}$ *be an* ω*-covering matrix on* λ*.*

CP($\mathcal{D}$) *holds if there is* A *unbounded subset of* λ *such that* $[A]^\omega$ *is covered by* $\mathcal{D}$*.*

$S(\mathcal{D})$ *holds if there is* S *stationary subet of* λ *such that for all families* $\{S_i : i < \omega\}$ *of stationary subsets of* S*, there are* $j < \omega$ *and* $\beta < \lambda$ *such that* $S_i \cap D(j, \beta)$ *is non-empty for all* $i < \omega$*.*

CP($\mathcal{D}$) is very close to what we are looking for.

We would like that $[A]^{<\lambda}$ is covered by $\mathcal{D}$ and not just $[A]^\omega$.

It is not transparent that $S(\mathcal{D})$ is of any use in solving our problem.

Regarding the consistency of these properties, we have the following results:

Fact 1. *Assume* MM. *Then* $S(\mathcal{D})$ *holds for all closed* ω*-covering matrices* $\mathcal{D}$ *on a regular* $\lambda > \aleph_1$.

Fact 2. *Assume* λ *is weakly compact. Then* $S(\mathcal{D})$ *holds for all closed,* ω*-covering matrices* $\mathcal{D}$ *on* λ.

Theorem 14. *Assume* PFA. *Then* $\mathrm{CP}(\mathcal{D})$ *holds for all* ω*-covering matrices* $\mathcal{D}$ *on a* $\lambda > \aleph_2$.

We give a proof of the first fact.

Dimostrazione. Let:

- $\mathcal{D}$ be a closed, ω-covering matrix on λ,
- $\{S_i : i < \omega\}$ be a family of stationary subsets of S^λ_ω.

By MM find $\delta \in S^\lambda_{\aleph_1}$ such that S_i reflects on δ for all $i < \omega$.

At least one $D(j, \delta)$ is unbounded in δ since $\bigcup_i D(i, \delta) = \delta$ and δ has cofinality ω_1.

Then $D(j, \delta)$ is a club subset of $S^\kappa_\omega \cap \delta$ since $\mathcal{D}$ is a closed matrix.

Thus $S_i \cap D(j, \delta)$ is non-empty for all $i < \omega$. $\square$

Regarding the mutual relation between CP(D) and $S(\mathcal{D})$:

Lemma 15. *The following holds:*

(i) CP($\mathcal{D}$) *implies* $S(\mathcal{D})$ *whenever* $\mathcal{D}$ *is closed,*

(ii) $S(\mathcal{D})$ *implies* CP($\mathcal{D}$) *whenever* $\mathcal{D}$ *is transitive.*

For the matrices that we are considering $S(\mathcal{D})$ and CP($\mathcal{D}$) are equivalent.

We prove item *(ii)* for an ω-covering matrix $\mathcal{D}$:

Assume $S(\mathcal{D})$ holds for a transitive ω-covering matrix $\mathcal{D}$ on λ. Let:

- S witness $S(\mathcal{D})$,
- T_i be the set of $\alpha \in S$ such that
$$S^i_\alpha = \{\beta \in S \setminus \alpha : \alpha \in D(i, \beta)\}$$
is stationary.

Now:

$$S \setminus \alpha = \bigcup \{S^i_\alpha : i < \omega\}$$

So for each α,

$S^{i_\alpha}_\alpha$ is stationary for some i_α.

Thus:

$$\lambda = \bigcup \{T_i : i < \omega\}$$

So T_i is stationary for some $i < \omega$.

Recall that

$$S^i_\alpha = \{\beta \in S \setminus \alpha : \alpha \in D(i, \beta)\}$$

and that

$$T_i = \{\alpha : S^i_\alpha \text{ is stationary}\}$$

Claim 16. $[T_i]^\omega$ *is covered by* $\mathcal{D}$ *for all* i*.*

Pick any $X \in [T_i]^\omega$ and consider the family of stationary sets $\{S^i_\alpha : \alpha \in X\}$.

By $S(\mathcal{D})$ there are some $j < \omega$ and $\delta < \lambda$ such that

$$S^i_\alpha \cap D(j, \delta) \neq \emptyset$$

for all $\alpha \in X$.

W.l.og. we can suppose that $j \geq i$.

For any $\alpha \in X \subseteq T_i$, there is $\beta_\alpha \in D(j, \delta) \cap S^i_\alpha$, i.e. β_α is such that $\alpha \in D(i, \beta_\alpha)$.

Since $\mathcal{D}$ is a transitive covering matrix and $j \geq i$, for all $\alpha \in X$:

$$\alpha \in D(i, \beta_\alpha) \subseteq D(j, \beta_\alpha) \subseteq D(j, \delta).$$

So $X \subseteq D(j, \delta)$.

X is arbitrary, so $[T_i]^\omega$ is covered by $\mathcal{D}$. $\square$

Fact 3. *Assume:*

- $\aleph_0 < \lambda < \aleph_\omega$.

- $\mathcal{D}$ *is an* ω*-covering matrix on* λ*,*

- T *is an unbounded subset of* λ *such that* $[T]^\omega$ *is covered by* $\mathcal{D}$*.*

Then $[T]^{<\lambda}$ *is covered by* $\mathcal{D}$*.*

In particular:

If $\mathcal{D}$ is an ω-covering matrix on $\aleph_2$ and $[A]^{\aleph_0}$ is covered by $\mathcal{D}$, $[A]^{\aleph_1}$ is covered by $\mathcal{D}$.

We have shown that under MM

$$S^{\aleph_{\omega+1}}_{>\aleph_1} \in \mathcal{I}[\aleph_{\omega+1}, \aleph_\omega],$$

since:

(a) For all $n > 1$, MM implies $S(\mathcal{D})$ for all closed ω-covering matrices $\mathcal{D}$ on $\aleph_n$.

(b) If $\mathcal{D}$ is transitive, $S(\mathcal{D})$ implies there is A unbounded in $\aleph_n$ such that $[A]^\omega$ is covered by $\mathcal{D}$.

(c) If A is unbounded in $\aleph_n$ and such that $[A]^\omega$ is covered by $\mathcal{D}$, then $[A]^{\aleph_{n-1}}$ is covered by $\mathcal{D}$.

This is what was needed to get that any point of cofinality $\aleph_n$ is d-approachable if $\mathcal{D}(d)$ is a normal, transitive and closed ω-covering matrix.

$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ **fails under** MM.

Assume not.

Take a structure M such that

$$\mathrm{otp}(M \cap \aleph_{\omega+1}) = \aleph_2$$

There is a normal, closed, transitive, ω-covering matrix $\mathcal{D} = \{D(i, \alpha) : \alpha < \aleph_{\omega+1}, i < \omega\} \in M$ with $\beta_{\mathcal{D}} = \aleph_\omega$.

Let $X = M \cap \aleph_{\omega+1}$ and $\mathrm{otp}(M \cap \aleph_\omega) = \delta < \aleph_2$.

Then for all $i < \omega$ and $\alpha \in M \cap \aleph_{\omega+1}$:

$$\mathrm{otp}(\overline{X} \cap D(i, \beta)) \leq \mathrm{otp}(X \cap D(i, \beta)) + 1 < \delta$$

since $\mathrm{otp}(\overline{Y}) \leq \mathrm{otp}(Y) + 1$ for any set of ordinals Y.

Let $\mathcal{E}$ be the transitive collapse of

$$\{\overline{X} \cap D(i, \beta) : i < \omega, \beta \in \overline{X}\}$$

Then $\mathcal{E}$ is a transitive, closed ω-covering matrix on $\aleph_2$ with $\beta_{\mathcal{E}} = \delta < \aleph_2$.

This is impossible since by MM there is H unbounded in $\aleph_2$ such that $[H]^{\aleph_1}$ is covered by $\mathcal{E}$.
$\square$

Some open problems

Is it consistent that $S^{\kappa^+}_{\aleph_2}$ *is not in* $\mathcal{I}[\kappa^+]$ *for a* κ *of countable cofinality?*

Does MM *decide whether* $S^{\aleph_{\omega^2+1}}_{\aleph_{\omega+1}}$ *is in* $\mathcal{I}[\aleph_{\omega^2+1}]$*?*