On the notion of guessing model

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0-guessing models are the image of elementary embeddings

Let $\gamma < \lambda$ be regular cardinals. Let $j : H_\gamma \to H_\lambda$ be a non trivial elementary embedding with

- $\overline{\kappa} = crit(j)$
- $\kappa = j(crit(j))$.

$M = j[H_\gamma]$ has the following crucial property:

Fact

For every $X \in M$ and every $d \in P(X)$, there is $z \in M$ such that $z \cap M = d \cap M$.

Proof.

$M = j[H_\gamma] \prec H_\lambda$ is isomorphic to H_γ:
If $X \in M$, $X = j(Y)$ for some $Y \in H_\gamma$.
If $d \in P(X)$, $d \cap M = d \cap j[Y]$.
Let $e = j^{-1}[d \cap M]$.
Then $e \in P(Y) \subseteq H_\gamma$.
Thus $j(e) \in j[H_\gamma] = M$.
It is clear that $j(e) \cap M = j[e] = d \cap M$.

□

Definition

Given some $N \prec V_\lambda$ and some $d \subseteq X$ for some $X \in N$, we say that d is *N-guessed* if $d \cap N = z \cap N$ for some $z \in N$.

Lemma (Magidor)

Assume $M \prec H_\lambda$ is such that for all $X \in M$ every $d \in P(X)$ is M-guessed. Then the transitive collapse of M is H_γ, where $\gamma = \mathrm{otp}(M \cap Ord)$.

Such models $M \prec H_\lambda$ give non-trivial elementary embeddings from the transitive collapse of M into H_λ!
We want to generalize this type of guessing properties of structures to a wider class of models.
The next example show the difficulty we may encounter.

Assume $M = \bigcup\{M_\alpha : \alpha < \omega_1\} \prec H(\aleph_3)$ is internally approachable of size $\aleph_1$.
Let $C = \{M_\alpha \cap \aleph_2 : \alpha < \omega_1\}$. Then C cannot be guessed. Why?
Otherwise C would be guessed by a $D \in M$ such that $D \cap M$ is unbounded in $\aleph_2 \cap M$.
Thus M models D is an unbounded subset of $\aleph_2$.
Thus $\omega_1 = \text{otp}(C) = \text{otp}(D \cap M) = M \cap \aleph_2 > \omega_1$.
Notice however that all the initial segments of C are in M.....

Conclusion:

An internally approachable model M of size $\aleph_1$ contains a subset $C \subseteq M \cap \aleph_2$ such that:

1. $C \cap X \in M$ for all countable $X \in M$,
2. C is not guessed, i.e. $C = C \cap M \neq E \cap M$ for all $E \in M$.

On the other hand:

> *If $M \prec H(\theta)$ has size $\aleph_1$, then for any set C which is M-guessed, item 1 above holds.*

QUESTION: Can there be an $M \prec H(\theta)$ of size $\aleph_1$ such that item 1 above is a sufficient condition for a set to be M-guessed?

Let $M \prec H_\lambda$ be a proper elementary substructure.
$\kappa_M = \min\{\alpha \in M : M \cap \alpha \neq \alpha\}$
$\overline{\kappa}_M = \sup(M \cap \kappa_M)$.

κ_M is always a regular cardinal!
In the previous examples:
If $M = j[H_\gamma]$, $\kappa_M = j(crit(j))$ is inaccessible as well as $\overline{\kappa_M}$,
If $M \prec H_{\aleph_3}$ contains ω_1 and has size ω_1, $\kappa_M = \aleph_2$ while $\overline{\kappa_M}$ is an ordinal of size $\aleph_1$.

Definition

Given $M \prec H_\lambda$, a cardinal $\delta \leq \kappa_M$, some $X \in M$ and some $d \in P(X)$, we say that:

d is (δ, M)-*approximated* if $d \cap z \in M$ for all $z \in M \cap P_\delta X$.

Definition

Let $M \prec H_\lambda$ and $X \in M$.
M is a *δ-guessing model for X* if every (δ, M)-approximated subset of X is M-guessed.
M is a *δ-guessing model* if M is a δ-guessing model for X for all $X \in M$.
M is a *guessing model* if M is a *δ-guessing model* for some $\delta \leq \kappa_M$.

Magidor's characterization of the image of an elementary embedding $j : H_\gamma \to H_\lambda$ now can be phrased:

Lemma (Magidor)

Assume $M \prec H_\lambda$ is a 0-guessing model. Then M is isomorphic to H_γ, where $\gamma = \mathrm{otp}(M \cap \lambda)$.

Notice that since any $M \prec H_\lambda$ is closed under finite sequence M is a 0-guessing model iff M is an $\aleph_0$-guessing model.

Can there be $\aleph_1$-guessing models M which are not $\aleph_0$-guessing models? Yes!

Theorem (Viale, Weiss)

Assume PFA. *Then for every $\theta \geq \aleph_2$ there are $\aleph_1$-guessing models $M \prec H_\theta$ with $\kappa_M = \aleph_2$.*

Why is this interesting? One reason is the following reformulation of a theorem of Magidor:

Theorem (Magidor)

κ is supercompact iff κ is inaccessible and for every $\theta \geq \kappa$ there is a guessing model $M \prec H_\theta$ with $\kappa_M = \kappa$.

Assuming PFA, $\aleph_2$ is "supercompact" apart from the missing inaccessibility.

This can be substantiated further.
For example we can use the two results above plus some other results to infer that all known methods to force PFA require the existence of a strongly compact cardinal in the ground model.
We can argue that in most of the known cases a *supercompact* is needed.
This is the content of the following paper by me and Weiss:
On the consistency strength of the proper forcing axiom
http://arxiv.org/abs/1012.2046

In the rest of this talk I want to motivate more why guessing models are interesting:

1. They allow for very simple and *modular* proofs of many combinatorial consequences of forcing axioms and supercompactness:
   The proofs are *modular* as they rely only of the assumption of the existence of a guessing model M, it is not relevant in the argument which $\delta \leq \kappa_M$ witnesses M is a guessing model.
2. Guessing models are very simple and "canonical" objects. For example the isomorphism type of a guessing model $M \prec H_\lambda$ is essentially determined by $\overline{\kappa}_M$ and $\text{otp}(M \cap Card)$.

I strongly believe that guessing models will be relevant in the search of a proof that PFA is equiconsistent with a supercompact cardinal.

As a sample of the first assertion:

Theorem (Viale,Weiss)

Assume there are stationarily many guessing models $M \prec H_\theta$ with $\kappa_M = \kappa$. Then $\square_\lambda$ fails for any $\lambda \geq \kappa_M$ such that $\lambda^+ < \theta$.

Lemma

Assume M is a guessing model. Then $M \cap Ord$ is closed under countable suprema.

Proof of the Lemma

Assume not for some δ-guessing model M. To simplify slightly the argument let's assume $\delta = \aleph_1$.
This is the only part of the proof where the argument is not modular and depends on δ.
This means we are interested in subset d of M which are $(\aleph_1, M)$-approximated, i.e. such that $d \cap z \in M$ for all countable $z \in M$.

Let $\xi \in M$ have uncountable cofinality be such that $\sup(M \cap \xi) \notin M$ has countable cofinality.
Then $M \cap [\sup(M \cap \xi), \xi)$ is empty.
Now for any $d \in M$ countable, $d \cap \xi$ is bounded below $\sup(M \cap \xi)$, since $d \subseteq M$. Else:
$\sup(M \cap \xi) \leq \sup(d \cap \xi) < \xi$ and $\sup(d \cap \xi) \in M$.
Fix in R, $d^* = \{\alpha_n : n \in \omega\} \subseteq M \cap \xi$ increasing and cofinal sequence converging to ξ.
Then $d^* \cap d \in R$ is finite for all countable $d \in M$ and thus belongs to M for all such d.
Thus d^* is an $(\aleph_1, M)$-approximated subset of M.
This means that $d^* = d^* \cap M = e \cap M$ for some $e \in M \cap P(\xi)$.
Now $M \models e$ is an unbounded subset of ξ, thus $\text{otp}(e) \geq \text{cf}(\xi)$, in particular $\text{otp}(e \cap M) \geq \text{otp}(\text{cf}(\xi) \cap M) > \omega = \text{otp}(d^*)$.
Thus $e \cap M \neq d^*$ which is the desired contradiction. □

Proof of the theorem

Assume not and pick a square sequence $\mathcal{C} = \{C_\alpha : \alpha < \lambda^+\}$.
Let $M \prec H_\theta$ be a δ-guessing model for some $\delta \leq \kappa_M$ such that $\mathcal{C} \in M$.
Let $\eta = \sup M \cap \lambda^+$.
Since M is a guessing model, M is closed under countable sequences.
Thus $\mathrm{cf}(\eta) > \aleph_0$.
Pick $\alpha \in M$ limit point of C_η. Then $C_\eta \cap \alpha = C_\alpha \in M$.
If $z \in M$ has size less than δ we can find $\alpha \in M$ limit point of C_η above $\sup(z \cap \lambda^+)$.
Then $C_\eta \cap z = C_\alpha \cap z \in M$.
Thus C_η is (M, δ)-approximated.
Since M is a guessing model there is $C \in M$ such that $C \cap M = C_\eta \cap M$.

Now $\mathrm{otp}(C_\alpha \cap M) \leq \mathrm{otp}(\lambda \cap M)$ for all $\alpha \in M$.
Observe that $C_\eta \cap M$ is the coherent union of $C_\alpha \cap M$ for $\alpha \in M$ limit point of C_η.
Thus $\mathrm{otp}(C_\eta \cap M) \leq \mathrm{otp}(\lambda \cap M)$.
On the other hand $C \in M$ is an unbounded subset of $\lambda^+ \cap M$. Thus $\mathrm{otp}(C \cap M) = \mathrm{otp}(M \cap \lambda^+)$.
We reached a contradiction since we assumed $C \cap M = C_\eta \cap M$ but the two sets have different order types. □

Why guessing models are canonical objects?

Assume λ is inaccessible and $M, N \prec H_\lambda$ are 0-guessing models.

Assume $\eta = \overline{\kappa}_M = \overline{\kappa}_N$.

Then $\text{otp}(M \cap \kappa_M^+) = \text{otp}(N \cap \kappa_N^+) = \eta^+$.

Notice that $\kappa_M \neq \kappa_N$ could be conceivable...

In general one can show:

Lemma

Assume $M, N \prec H_\lambda$ are 0-guessing models. If $\text{otp}(M \cap \mathit{Card}) = \text{otp}(N \cap \mathit{Card})$ and $\overline{\kappa}_M = \overline{\kappa}_N$, then M and N are isomorphic.

What can be said about isomorphism types of arbitrary δ-guessing models?

Theorem (Viale, Weiss)

Assume PFA. *Then for every* $\theta \geq \aleph_2$ *there are stationarily many* $\aleph_1$*-guessing models* $M \prec H_\theta$ *such that:*

- $\kappa_M = \aleph_2$,
- *M is internally club i.e* $M \cap P_{\omega_1} M$ *is a club subset of* $P_{\omega_1} M$.

Theorem (Viale)

Assume $M, N \prec H_\lambda$ *are internally club* $\aleph_1$*-guessing models such that:*

- $\overline{\kappa}_M = \overline{\kappa}_N$,
- $2^{\aleph_0} = \kappa_M = \kappa_N$,
- $\text{otp}(M \cap Card) = \text{otp}(N \cap Card)$.

Then M and N are isomorphic structures.

This is not fully satisfactory since for example it is not known whether the following is possible under PFA:

> *There are two internally club and $\aleph_1$-guessing models $M, N \prec H_\lambda$ such that:*
>
> - *for some regular $\theta \in M \cap N$, $\sup(M \cap \theta) = \sup(N \cap \theta)$,*
> - *$M \cong N$,*
> - *$M \cap \theta \neq N \cap \theta$.*

This cannot be for 0-guessing models due to the following:

Lemma

Assume $M, N \prec H_\lambda$ are 0-guessing models such that $\sup(M \cap \theta) = \sup(N \cap \theta) = \eta$ for some regular cardinal $\theta \in M \cap N$. Then $M \cap \theta = N \cap \theta$.

Proof of the lemma

Pick a partition $\{S_\alpha : \alpha < \theta\} \in M \cap N$ of θ in θ-many stationary sets. Observe that due to the fact that M is a 0-guessing model, the following holds:

> *For any $S \in M \cap P(\theta)$ set of points of countable cofinality, S is stationary iff S reflects on $\sup(M \cap \theta) = \eta$.*

Thus:

$$M \cap \theta = \{\alpha : S_\alpha \text{ reflects on } \eta\} = N \cap \theta.$$

□

What the above lemma shows is the following:

Given a stationary set $\mathcal{S}$ of $M \prec H_\lambda$ let:

$$T_{\mathcal{S}} = \{M \cap Card \cap \theta : M \in \mathcal{S}, \theta \in \lambda^+ \cap Card\}$$

If $\mathcal{S}$ is the set of $M \prec H_\lambda$ which are 0-guessing models, $T_{\mathcal{S}}$ is tree ordered by end extension.

Can we isolate under forcing axioms a family of guessing models with this same property?
Yes if we use MM.

Given a set of ordinals S such that S is a stationary subset of $\sup(S)$ let:

$$P^*(S) = \{T \subseteq S : T \text{ is stationary in } \sup(S)\}.$$

Definition

$M \prec H_\theta$ is an *S-faithful model* if for all $T \in P(S) \cap M$,

T reflects on $\sup(M \cap S)$ *iff $T \in P^*(S)$.*

$M \prec H_\theta$ is a *λ-faithful model* if M is $E_\lambda^{\aleph_0}$-faithful.
$M \prec H_\theta$ is a *faithful model* if M is $E_\lambda^{\aleph_0}$-faithful for all regular $\lambda \in M$.

The following lemma motivates the definition of faithful models:

Lemma

Assume $M_0, M_1 \prec H_\theta$ are λ-faithful models for some regular $\lambda \in M_0 \cap M_1$ and $\sup(M_0 \cap \lambda) = \sup(M_1 \cap \lambda)$. *Then $M_0 \cap \lambda = M_1 \cap \lambda$.*

Lemma

Assume MM. *Then there for every* $\lambda \geq \omega_1$ *there are stationarily many* $M \prec H_\lambda$ *such that:*

- $\kappa_M = \aleph_2$,
- M *is faithful,*
- M *is* $\aleph_1$*-guessing,*
- M *is internally club.*

If $\mathcal{S}$ is the set of the above models
$T_{\mathcal{S}}$ is a tree ordered by end extension.

SUMMING UP:

There are two main properties of supercompactness which are used to get many of its interesting combinatorial consequences:

1. Existence of guessing models with $\kappa_M = \kappa$: this in conjunction with inaccessibility characterizes a supercompact cardinal κ and holds on $\aleph_2$ assuming PFA.
2. Existence of faithful models (i.e. models computing correctly stationarity of their members). This follows from supercompactness and to some extent from strong compactness, it follows from MM but not from PFA. It is not clear to me whether this property characterizes supercompactness.

It has to be understood to what extent the existence of faithful and guessing models given by MM can be transferred to inner transitive classes. Apart from the result appeared in our joint paper I and Christoph Weiss have several partial results which are still too partial to be presented in a talk but which might be of interest.

Much of what has been presented here can be found in my paper *On the notion of guessing model*, available at:
http://arxiv.org/abs/1012.2212

THANK YOU FOR YOUR ATTENTION AND BEST WISHES OF A HAPPY NEW YEAR