Forcing and absoluteness as means to prove theorems

Matteo Viale

Dipartimento di Matematica
Università di Torino

21 November 2011
Paris

FORCING

Forcing was introduced in 1963 by Paul Cohen to prove the independence with respect to ZFC of the continuum hypothesis, the first in the list of 23 Hilbert's problems.
It soon emerged that forcing is a very powerful tool to prove independence results in ZFC and also in many other branches of pure mathematics, for example:

- Group theory (Whitehead's problem) Shelah 1974,
- General topology: Todorčević and Moore's results on the S-space and the L-space problems,
- Functional analysis: many results of Todorčević on Banach spaces,
- Operator algebras: Farah's works on the automorphisms of the Calkin algebra which develops on Shelah and Veličković's analysis of the automorphism group of $P(\omega)/\text{FIN}$,
- ....

I'm surely forgetting many fundamental results and contributions, hopefully by people not in this room.

FORCING

Forcing can be seen as a “computable” function:

$$(M, B) \mapsto M^B$$

- M is a (transitive) model of (a large enough fragment of) ZFC.
- $B \in M$ is a boolean algebra.
- M^B is a Boolean valued model of ZFC and is a definable class in M.
- There is a definable injective map $\check{i}_B : M \to M^B$ such that M is naturally identified with a definable subclass of M^B.
- Truth in M^B depends on
  - The first order theory of M.
  - The combinatorial properties that M gives to B.
- The truth predicate for M^B is (almost) definable in M with parameter B.

The forcing relation

To evaluate the semantics of M^B one introduces the forcing relation:

$$p \Vdash_B \phi(\tau_1, \ldots, \tau_n)$$

$p \in B$, $\tau_1, \ldots, \tau_n \in M^B$ and $\phi(x_1, \ldots, x_n)$ is a formula in the language of set theory.
The relation $p \Vdash_B \phi(\tau_1, \ldots, \tau_n)$ is definable in M in the parameters $p, \tau_1, \ldots, \tau_n$ by recursion on the logical complexity of ϕ:

- If $p \Vdash_B \phi$ and $q \leq p$, then $q \Vdash_B \phi$,
- $p \Vdash_B \phi \wedge \psi$ if $p \Vdash_B \phi$ and $p \Vdash_B \psi$,
- $p \Vdash_B \neg\phi$ if $q \nVdash_B \phi$ for all $q \leq p$,
- $p \Vdash_B \forall x \phi(x)$ if $p \Vdash_B \phi(\tau)$ for all $\tau \in M^B$.

What is difficult is to define $p \Vdash_B \tau_1 \in \tau_2$ and $p \Vdash_B \tau_1 = \tau_2$.
All the novelty of forcing is hidden in the truth clause for negation which depends on the combinatorial properties of the partial order B.

What holds in M^B

Theorem (Cohen)

Assume M is a (transitive) model of ZFC, $B \in M$ is a boolean algebra and $p \in B$. Then for any axiom ϕ of ZFC, $p \Vdash_B \phi$.

What else holds in M^B depends on the choice of B and the first order properties of M.

Baire's category theorem and forcing

Given a boolean algebra B,
G is a filter on B if:

- if $p \in G$ and $q \geq p$, $q \in G$,
- $0_B \notin G$, $1_B \in G$,
- if $p, q \in G$, $p \wedge q \in G$.

D is dense in B if for all $p \in P$, there is $q \in D$ such that $q \leq p$.
A is open in B if whenever $p \in A$ and $q \leq p$, $q \in A$ as well.

Baire's category theorem and forcing

$\mathsf{FA}_\lambda(B)$ holds if for all family $\{D_\xi : \xi < \gamma\}$, $(\gamma < \lambda)$ of dense open subsets of B, there is a filter G which meets all these dense sets.

Theorem (Baire's category theorem)

For all boolean algebras B, $\mathsf{FA}_{\aleph_1}(B)$ holds.

Towards Cohen's forcing Theorem

From now on, we assume V exists and is the "true" universe of sets. Else (if one does not want to be platonist) one has to reformulate everything with more care.

Let $(M, \in) \in V$ be a model of a large enough fragment of ZFC.
Typically:

- $M \prec H_\lambda$ or $M \prec V_\alpha$ for some large enough regular cardinal λ or some big enough ordinal α,
- $M = \pi_N[N]$ where $N \prec H_\lambda$ ($N \prec V_\alpha$) and π_N is the transitive collapse of the structure $(N, \in \cap N^2)$.

Let $B \in M$ be such that M models B is a boolean algebra.

Definition

G is an M-generic filter if:
$G \cap D \cap M$ is non-empty for all $D \in M$ dense open subset of B.

Cohen's forcing Theorem

Theorem (Cohen's forcing Theorem)

Assume:

- $M \in V$ *is a transitive model of* ZFC *of size less than* λ*,*
- $B \in M$ *is a boolean algebra such that* $\mathrm{FA}_\lambda(B)$ *holds in* V*,*
- $G \in V$ *is an* M*-generic filter.*

Then there is a surjective map $\sigma_G : M^B \to M[G]$ *such that:*

- $M[G]$ *is transitive.*
- $\sigma_G \circ \check{i}_B(a) = a$ *for all* $a \in M$*.*
- $(M[G], \in) \models \phi(\sigma_G(\tau_1), \ldots, \sigma_G(\tau_n))$ *if and only if there is* $p \in G$ *such that*

$$(M, \in) \models [p \Vdash_B \phi(\tau_1, \ldots, \tau_n)].$$

Cohen's absoluteness Lemma

Corollary

Assume that:

- $\phi(x, r)$ *is a* Δ_0*-formula with real parameter* r*.*
- $B \in V$ *is a Boolean algebra such that* $1_B \Vdash_B \exists x \phi(x, \check{i}_B(r))$*.*

Then $L(\mathbb{R}) \models \exists x \phi(x, r)$*.*

Proof:

Assume $B \in V$ is a Boolean algebra such that

$$1_B \Vdash_B \exists x \phi(x, \check{i}_B(r)).$$

To simplify matters assume there is an inaccessible λ such that $B \in V_\lambda$ (redundant assumption).
Then $V_\lambda \models$ ZFC and

$$V_\lambda \models [1_B \Vdash_B \exists x \phi(x, \check{i}_B(r))].$$

Pick $N \prec V_\lambda$ countable such that $B \in N$.
Let $M = \pi_N[N]$ and $Q = \pi_N(B)$. Notice that $r \in P(\omega)$ and $\pi_N(\omega) = \omega$,
Thus $\pi_N(r) = r$.

Proof continued

Since $\pi_N : N \to M$ is an isomorphism and $Q = \pi_N(B)$, $\pi_N(r) = r$,

$$M \models [1_Q \Vdash_Q \exists x \phi(x, \check{i}_Q(r))]$$

Now M is countable and transitive, $Q \in M$ and $\mathsf{FA}_{\aleph_1}(Q)$ holds in V.
Thus there is $G \in V$ which is an M-generic filter for Q.

Proof continued

By Cohen's forcing Theorem we can define $\sigma_G : M^Q \to M[G]$ surjective such that

- $\sigma_G \circ \check{i}_Q(a) = a$ for all $a \in M$,
- $M[G]$ is transitive,
- $M[G] \models \psi$ if there is $p \in G$ such that

$$M \models [p \Vdash_Q \psi].$$

In particular since $\sigma_G \circ \check{i}(r) = r$

$$M[G] \models \exists x \phi(x, r).$$

Proof continued

Thus there is $a \in M[G]$ such that $M[G] \models \phi(a, r)$.
Since $M[G]$ is countable and transitive, $M[G] \in L(\mathbb{R})$ and $M[G] \subset L(\mathbb{R})$, thus $a, r \in L(\mathbb{R})$.
Since $\phi(a, r)$ is a Σ_0-formula with parameters in $M[G] \subset L(\mathbb{R})$:

$$M[G] \models \phi(a, r) \iff L(\mathbb{R}) \models \phi(a, r).$$

In particular a witnesses that $L(\mathbb{R}) \models \exists x \phi(x, r)$. $\square$

Cohen's absoluteness Lemma reformulated

Actually if one doesn't want to commit to any philosophical position on the onthology of sets Cohen's absoluteness Lemma can be formulated as follows:

Corollary (Cohen)

Let T be any first order theory which extends ZFC *and $\phi(x, r)$ be a Σ_0 formula with a parameter r such that $T \vdash r \subseteq \omega$. TFAE:*

- $T \vdash \exists x \phi(x, r)$.
- $T \vdash$ *There exists a boolean algebra B such that $1_B \Vdash_B \exists x \phi(x, \check{i}_B(r))$.*

Woodin's absoluteness

Theorem (Woodin)

Assume there are class many Woodin cardinals in V, then for every formula ϕ with real parameters:

$$L(\mathbb{R})^V \models \phi$$

if and only if there exists a boolean algebra $B \in V$ such that

$$1_B \Vdash_B \phi^{L(\mathbb{R})}$$

Notice that we had to relativize the formulas to $L(\mathbb{R})$ to obtain the absoluteness results.
This is an unavoidable consequence of the fact that formulas which are not Σ_0 are neither upward absolute nor downward absolute between transitive structures.
(We needed the upward absoluteness of $\phi(a, r)$ to conclude the proof of Cohen's absoluteness.)

Effects of large cardinals on the theory of $L(\mathbb{R})$

If one investigates with care Woodin's proof, the assumption that V is transitive is redundant.
In particular Woodin actually proved:

Theorem (Woodin)

Let T be any theory which extends ZFC+*there are class many Woodin cardinals. Then for any formula ϕ TFAE:*

- $T \vdash [L(\mathbb{R}) \models \phi]$,
- $T \vdash$ *There is a boolean algebra B such that* $1_B \Vdash_B \phi^{L(\mathbb{R})}$.

What can be said about $L(P(\lambda))$ and forcing absoluteness?

If $\lambda \geq \aleph_2$ there are (provably in ZFC) boolean algebras B such that $\mathrm{FA}_\lambda(B)$ is false.
Nonetheless we can prove on the same lines the following:

Corollary (Generalized Cohen's absoluteness)

Assume that:

- $\phi(x, p)$ *is a* Δ_0*-formula with parameter* $p \in P(\lambda)$.
- $B \in V$ *is a Boolean algebra such that* $1_B \Vdash_B \exists x \phi(x, \check{i}_B(p))$.
- $\mathrm{FA}_{\lambda^+}(B)$ *holds in* V.

Then $L(P(\lambda)) \models \exists x \phi(x, p)$.

When does some B satisfy FA(λ)?

Without further assumptions, the generalized version of Cohen's absoluteness lemma is not so useful.
Whether a boolean algebra $B \in V$ satisfy $\mathsf{FA}_\lambda(B)$ for some $\lambda \geq \omega_2$ is very much dependent on the particular choice of set theoretic universe V we work in.
We need more assumption on V.
We have a very natural candidate for such an assumption, subsumed by the following slogan:

> $\mathsf{FA}_\lambda(B)$ *holds for all B for which we cannot prove that it fails.*

Notice that for $\lambda = \aleph_1$ this slogan is trivially true:
in ZFC we can prove that $\mathsf{FA}_{\aleph_1}(B)$ holds for all B.

Martin's maximum makes the slogan true for $\lambda = \aleph_2$

A complete Boolean algebra B is locally stationary set preserving if there is some fixed $p \in B$ such that

$$p \Vdash_B \check{i}_B(S) \text{ is stationary in } \check{i}(\omega_1).$$

for every S which is a stationary subset of ω_1.
Shelah proved that if B is not locally stationary set preserving then $\mathrm{FA}_{\aleph_2}(B)$ fails.

Martin's maximum makes the slogan true for $\lambda = \aleph_2$, continued

Foreman, Magidor, Shelah proved the following:

Theorem

Assume there is a supercompact cardinal in V. Then there is a boolean algebra $\mathbb{B} \in V$ such that

$1_{\mathbb{B}} \Vdash_{\mathbb{B}} \mathsf{FA}_{\aleph_2}(B)$ *holds for all B which are locally stationary set preserving.*

Let us abbreviate by MM (Martin's maximum) the assertion that $\mathsf{FA}_{\aleph_2}(B)$ *holds for all B which are locally stationary set preserving*.

Why MM (actually MM^{++}) is useful.

The model $V^{\mathbb{B}}$ produced by Foreman, Magidor, Shelah prove the consistency of an enhanced version of MM which we call MM^{++}.
MM^{++} is a natural statement stronger than MM but not so simple to formulate.
MM (MM^{++}) is extremely powerful, a variety of problems in set theory and in general mathematics have a solution if we assume MM^{++} (often much less suffice):

- the continuum problem,
- the singular cardinal problem,
- the Suslin problem,
- all mathematical problems mentioned in the first slides,
- many, many other problems

have a solution assuming MM. Why?

Because of forcing absoluteness!

Theorem (V.)

Assume T is a theory which extends $\mathsf{ZFC} + \mathsf{MM}^{++} +$ *there are class many Woodin cardinals. For any π_2-formula $\phi(p)$ in a parameter p such that $T \vdash p \subset \omega_1$ the following are equivalent:*

- $T \vdash \phi^{H_{\omega_2}}(p)$,
- $T \vdash$ *There is a boolean algebra B which is stationary set preserving and such that* $1_B \Vdash \mathsf{BMM}$ *and* $1_B \Vdash_B \phi^{H_{\omega_2}}$.

Conclusions

First conclusion: we cannot use stationary set preserving forcings to prove independence results for the Π_2-theory of H_{ω_2} in models of $\mathsf{MM}^{++}+$ *large cardinals*.
We can just use these forcings to prove implications from MM^{++}.
Eventually we could use forcings which are not stationary set preserving to prove independence results for the theory $\mathsf{MM}^{++}+$ *large cardinals*.
It is not known if this can be the case, one line of attack could be to investigate whether Woodin's axiom $(*)$ is independent of MM^{++}.

Second conclusion: the above theorem makes MM^{++} a very appealing axiom to extend ZFC in conjunction with large cardinals:
it is simple to state and assert a maximality principle for the cardinal $\aleph_2$ which provably holds at $\aleph_1$ (i.e. that an enhanced version of the Baire's category theorem $\mathsf{FA}_{\aleph_2}(B)$ holds for the largest conceivable class of boolean algebras B).

Third conclusion: to make this axiom more appealing we should show that this maximality property does not hold just for $\aleph_1$ and $\aleph_2$ but can be formulated and proved consistent for all regular uncountable cardinals λ.

This is what I'm currently working on...

I should also say that I cut out many other absoluteness results of Woodin about CH and on axiom $(*)$, a variant of a weak version of MM which is independent of MM but not yet known to be independent of MM^{++}.

Thanks for your patience and attention.