Martin's maximum revisited: Woodin cardinals, forcing axioms, Ω-logic, continued.....

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....and partly repeated......

Preliminaries: complete embeddings of partial orders

For a poset Q and $q \in Q$,
$Q \restriction q$ denote the poset Q restricted to conditions $r \in Q$ which are below q
$\mathbb{B}(Q)$ denote its boolean completion, i.e. the complete boolean algebra of regular open subsets of Q
$Q \restriction q$ is naturally identified with a dense subset of $\mathbb{B}(Q \restriction q)$.

- P completely embeds into Q if there is an homomorphism $i : P \to \mathbb{B}(Q)$ which preseves the order relation and maps maximal antichains of P into maximal antichains of $\mathbb{B}(Q)$.
  With abuse of notation we shall call a complete embedding of P into Q any such homomorphism $i : P \to \mathbb{B}(Q)$.
- $i : P \to \mathbb{B}(Q)$ is a locally complete embedding (or a regular embedding) if $i : P \to \mathbb{B}(Q \restriction q)$ is a complete embedding for some $q \in Q$.

Lemma

The following are equivalent:

1. *P completely embeds into Q,*
2. *for any V-generic filter G for Q there is in $V[G]$ a V-generic filter H for P,*
3. *For some $p \in P$ there is a homomorphism $i : \mathbb{B}(P \restriction p) \to \mathbb{B}(Q)$ of complete boolean algebras.*

Proof

1 implies 2

If $i : P \to \mathbb{B}(Q)$ is a complete embedding and G is a V-generic filter for $\mathbb{B}(Q)$, then $H = i^{-1}[G]$ is a V-generic filter for P.

2 implies 1

Let $\dot{H} \in V^{\mathbb{B}(Q)}$ be a name such that

$$\Vdash_{\mathbb{B}(Q)} \dot{H} \text{ is a } V\text{-generic filter for } P$$

The map $p \mapsto \|\check{p} \in \dot{H}\|_{\mathbb{B}(Q)}$ is the desired complete embedding of P into Q.

$\square$

Quotient forcings

Given a complete embedding $i : P \to \mathbb{B}(Q \restriction q)$ let $Q/i[P]$ be the canonical V^P-name for the quotient forcing so that $P * Q/i[P]$ is forcing equivalent to $Q \restriction q$.

The quotient $P * Q/i[P]$ depends on P, i, Q and q. For example the following is possible:

- There is a P-name $\dot{S}$ which is forced by P to be a stationary subset of ω_1.
- There are $i_0 : P \to \mathbb{B}(Q \upharpoonright q_0)$, $i_1 : P \to \mathbb{B}(Q \upharpoonright q_1)$ distinct locally complete embeddings of P into Q such that if G_j is V-generic for $\mathbb{B}(Q \upharpoonright q_j)$ and $H_j = i_j^{-1}[G_j]$, then
  - $\sigma_{H_0}(\dot{S})$ is stationary in $V[G_0]$,
  - $\sigma_{H_1}(\dot{S})$ is stationary in $V[H_1]$ but non-stationary in $V[G_1]$.

In particular

$$\Vdash_P \dot{S} \text{ remains stationary in } V^{Q/i_0[P]}$$

$$\Vdash_P \dot{S} \text{ is stationary in } V^P \text{ and non stationary in } V^{Q/i_1[P]}$$

Preliminaries: stationary sets and the stationary tower forcing

S is stationary if for all $f : P_\omega(\cup S) \to (\cup S)$ there is an $X \in S$ such that $f[P_\omega(X)] \subset X$.
For a stationary set S and a set X, if $\cup S \subseteq X$ we let $S^X = \{M \in P(X) : M \cap \cup S \in S\}$, if $\cup S \supseteq X$ we let $S \restriction X = \{M \cap X : M \in S\}$.
If S and T are stationary sets we say that S and T are compatible if

$$S^{(\bigcup S) \cup (\bigcup T)} \cap T^{(\bigcup S) \cup (\bigcup T)}$$

is stationary.
For a set M we let $\pi_M : M \to V$ denote the transitive collapse of the structure $(M, \in)$ onto a transitive set $\pi_M[M]$ and we let $j_M = \pi_M^{-1}$.

$$R_\lambda = \{X : X \cap \lambda \in \lambda \text{ and } |X| < \lambda\}$$

$\mathbb{R}^{\lambda}_{\delta}$ is the stationary tower whose elements are stationary sets $S \in V_\delta$ such that

- $S \subset R_\lambda$
- $S \leq T$ if $\cup(T) \subset \cup(S)$ and $T^{\cup(S)} \supseteq S$.

$\mathbb{R}_\delta$ will denote $\mathbb{R}^{\aleph_2}_{\delta}$.
If G is V-generic for $\mathbb{R}^{\lambda}_{\delta}$, then G induces in a natural way a direct limit ultrapower embedding $j_G : V \to M^G$.
Elements of M^G are equivalence classes $[f]_G$ where $f : S \to V$ in V is a function with $S \in V_\delta$ a stationary subset of R_λ.
The identifications are given by the standard procedure for direct limits ultrapowers.

Theorem (Woodin)

Assume δ is a Woodin cardinal, $\lambda < \delta$ is regular and G is V-generic for $\mathbb{R}^{\lambda}_{\delta}$. Then

1. *M^G is a definable class in $V[G]$ and*
$$V[G] \models (M^G)^{<\delta} \subseteq M^G.$$
2. *$V_\delta, G \subseteq M^G$ and $j_G(\lambda) = \delta$.*
3. *$M^G \models \phi([f_1]_G, \ldots, [f_n]_G)$ if and only for some $\alpha < \delta$ such that $f_i : P(X_i) \to V$ are such that $X_i \in V_\alpha$ for all $i \leq n$:*
$$\{M \prec V_\alpha : V \models \phi(f_1(M \cap X_1), \ldots, f_n(M \cap X_n))\} \in G.$$

In particular

$$(H_{j_G(\lambda)})^{M^G} = V_\delta[G] = (H_\delta)^{V[G]}.$$

Moreover for every $\alpha < \delta$ and every set $X \in V_\alpha$,
$X = [\langle \pi_M(X) : M \prec V_\alpha, X \in M \rangle]_G$.

Preliminaries: Woodin cardinals are forcing axioms

Let $\mathbb{P}$ be a partial order.

- $\mathsf{MA}_{<\lambda}(\mathbb{P})$ holds if for all families of less than λ-many dense subsets of $\mathbb{P}$ there is a filter $G \subset \mathbb{P}$ which meets all the dense sets in the family.
- Given a structure M, G is an M-generic filter for $\mathbb{P}$ if $G \cap D \cap M$ is non-empty for all dense $D \subset \mathbb{P}$ which are in M.
- Ω_λ is the class of posets which regularly embeds into $\mathbb{R}_\delta^\lambda$ for some Woodin cardinal δ.

Theorem (Woodin)

Assume there are class many Woodin cardinals and λ is a successor cardinal. Then the following are equivalent:

1. $\mathsf{MA}_{<\lambda}(\mathbb{P})$ *holds*
2. $S^{\lambda}_{\mathbb{P}}$ *is stationary, where*

$$S^{\lambda}_{\mathbb{P}} = \{M \prec H_{|\mathbb{P}|^+} : M \in R_\lambda \text{ and there is an } M\text{-generic filter } H_M \text{ for } \mathbb{P}\}.$$

3. $\mathbb{P}$ *is in* Ω_λ.

The theorem should hold for all regular λ, not just for successor. However for the case λ inaccessible but not Woodin, I have not checked in all details whether Woodin's proof of this Theorem (Theorem 2.53 in the "first big book") generalizes.

Proof sketch, 3 implies 1:

If δ is Woodin then there is plenty of self-generic models $M \prec V_{\delta+1}$ for $\mathbb{R}^{\lambda}_{\delta}$, i.e.:

$$S_\delta = \{M \prec V_{\delta+1} : M \in R_\lambda \text{ and } \forall D \in M \text{ dense subset of } \mathbb{R}^{\lambda}_{\delta} \\ \exists S \in D \cap M \text{ such that } M \cap \cup S \in S\}$$

is stationary and is compatible with any condition T in $\mathbb{R}^{\lambda}_{\delta}$.
In particular if $\mathbb{P}$ completely embeds into $\mathbb{R}^{\lambda}_{\delta} \restriction T$, then for any $M \in S_\delta \cap T^{V_{\delta+1}}$ there is an M-generic filter for $\mathbb{P}$.

2 implies 3:

Assume

- G is V-generic for $\mathbb{R}^{\lambda}_{\delta}$,
- $H = \langle H_M : M \in S^{\lambda}_{\mathbb{P}} \rangle$ is such that H_M is $\pi_M[M]$-generic for $\pi_M(\mathbb{P})$,

then $[H]_G \in M^G$ is V-generic for $\mathbb{P}$.

SSP denote the class of posets which preserve stationary subsets of ω_1. Martin's maximum MM asserts that $\text{MA}_{<\omega_2}(\mathbb{P})$ holds for all $\mathbb{P} \in \text{SSP}$. The following sums up the current state of affair regarding the classes Ω_λ.

Theorem

Assume there are class many Woodin cardinals. Then:

1. $\Omega_{\aleph_1}$ *is the class of all posets (Woodin),*
2. $\text{SSP} \supseteq \Omega_{\aleph_2}$,
3. $\text{SSP} = \Omega_{\aleph_2}$ (MM) *is consistent relative to the existence of a supercompact cardinal (Foreman, Magidor, Shelah),*

Proof sketch

Proof of 1:
$S_{\mathbb{P}}^{\aleph_1}$ is a club subset of $P_{\omega_1}(H_{|\mathbb{P}|^+})$, since M-generics exist for all countable structures M such that $\mathbb{P} \in M$.
If G is V-generic for $\mathbb{R}_\delta^{\aleph_1}$ $[\langle H_M : M \in S_{\mathbb{P}}^{\aleph_1} \rangle]_G \in M^G \subset V[G]$ is V-generic for $\mathbb{P}$ where H_M is $\pi_M[M]$-generic for $\pi_M(\mathbb{P})$.
Proof of 2:
Check that $\mathbb{R}_\delta^{\aleph_2}$ is stationary set preserving for any Woodin cardinal δ.
Proof of 3:
Observe that MM holds only if $S_{\mathbb{P}}^{\aleph_2}$ is stationary for any stationary set preserving poset $\mathbb{P}$.

Ω-logic and Ω_λ-logic

Definition

Let $\phi(p)$ be a formula with parameters in the language of set theory enriched by terms for the elements of V and Γ be a class of posets. ϕ is Γ-consistent if $\|\phi(\check{p})\|_{\mathbb{B}} = 1_{\mathbb{B}}$ for some boolean algebra $\mathbb{B} \in \Gamma$.

Observe that Ω-logic is defined by $\Omega_{\aleph_1}$-consistency.

Ω_λ-logic is Ω_λ-invariant:

Theorem (Woodin)

Assume there are class many Woodin cardinals and $\mathbb{P} \in \Omega_\lambda$. Let $\phi(p)$ be a first order formula with parameters in H_λ. Then

$$(\phi(p) \text{ is } \Omega_\lambda\text{-consistent})^V$$

if and only if

$$(\phi(p) \text{ is } \Omega_\lambda\text{-consistent})^{V^{\mathbb{P}}}$$

Why we shouldn't neglect Ω_λ-logic

Observe that Woodin's characterization of forcing axioms gives the following outcome when we go to their bounded versions:

Theorem

Assume there are class many Woodin cardinals. Let λ be a regular cardinal and $\phi(y, \vec{p})$ be a Δ_1-property in the parameter $\vec{p} \in H_\lambda$. The following are equivalent:

1. $H_\lambda \models \exists y \phi(y, \vec{p})$
2. *There is $\mathbb{P} \in \Omega_\lambda$ such that*

$$\Vdash_{\mathbb{P}} \exists y \phi(y, \vec{p})$$

For Σ_1-formulas with parameters in H_λ, Ω_λ-consistency overlaps with truth!!!

MM^{++} can be the right axiom for $\Omega_{\aleph_2}$-logic.

Definition

MM^{++} holds if $T_{\mathbb{P}}$ is stationary for all $\mathbb{P} \in \mathsf{SSP}$, where $M \in T_{\mathbb{P}}$ iff

- $M \prec H_{|\mathbb{P}|^+}$ is in $R_{\aleph_2}$,
- there is an M-generic filter H for $\mathbb{P}$ such that $\sigma_H(\dot{S})$ is stationary for all $\dot{S} \in M$ $\mathbb{P}$-name for a stationary subset of ω_1.

Theorem (Foreman, Magidor, Shelah)

Assume κ is supercompact, $f : \kappa \to V_\kappa$ is a Laver function and

$$\{(P_\alpha, \dot{Q}_\alpha) : \alpha \leq \kappa\}$$

is a revised countable support iteration such that

- $P_\alpha \Vdash \dot{Q}_\alpha$ *is semiproper,*
- $P_{\alpha+1} \Vdash |\alpha| = \aleph_1$,
- $\dot{Q}_\alpha = f(\alpha)$ *if* $P_\alpha \Vdash f(\alpha)$ *is semiproper.*

Let G be V-generic for P_κ. Then MM^{++} *holds in* $V[G]$.

Theorem

Assume there are class many Woodin cardinals. Then the following are equivalent:

1. MM^{++} *holds.*
2. *For every Woodin cardinal* δ *and every stationary set preserving poset* $\mathbb{P} \in V_\delta$ *there is* $T \in \mathbb{R}_\delta^{\aleph_2}$ *and a complete embedding* i *of* $\mathbb{P}$ *into* $\mathbb{Q} = \mathbb{R}_\delta^{\aleph_2} \restriction T$ *such that*

$$\Vdash_{\mathbb{P}} \mathbb{Q}/i[\mathbb{P}] \text{ is stationary set preserving}$$

Proof sketch, 1 implies 2:

For all $M \in T_{\mathbb{P}}$, fix H_M M-generic such that $\sigma_{H_M}(\dot{S})$ is a stationary subset of ω_1 for all names $\dot{S} \in M$ for stationary subsets of ω_1.
Assume G is V-generic for $\mathbb{R}_{\delta}^{\aleph_2}$ with $T_{\mathbb{P}} \in G$.
Then

$$H = [\langle \pi_M[H_M] : M \in T_{\mathbb{P}} \rangle] \in M^G \subset V[G]$$

is V-generic for $\mathbb{P}$.

Let $\dot{H}$ be the canonical name in $\mathbb{Q} = \mathbb{R}_\delta \restriction T_\mathbb{P}$ so that $\sigma_G(\dot{H}) = H$, then

$$\begin{aligned} i : & \mathbb{P} \to \mathbb{B}(\mathbb{Q}) \\ & p \mapsto \|p \in \dot{H}\|_{\mathbb{B}(\mathbb{Q})} \end{aligned}$$

is a complete embedding of $\mathbb{P}$ into $\mathbb{Q}$ and

$\Vdash_\mathbb{P} \mathbb{Q}/i[\mathbb{P}]$ is stationary set preserving.

Main result

Theorem

Assume MM^{++} *holds in* V *and there are class many Woodin cardinals. Then the* Π_2*-theory of* $H_{\aleph_2}$ *cannot be changed by stationary set preserving forcings which preserve* BMM.

Larson has shown that $\mathrm{MM}^{+\omega}$ is compatible with the existence of a well-order of H_{ω_2} definable in H_{ω_2} without parameters. It is open whether this is compatible with MM^{++}, moreover:

Question

Assume MM^{++} and large cardinals of any sort exists in V. Is $L(P(\omega_1))$ a $\mathbb{P}_{\max}$-extension of $L(\mathbb{R})$?

Questions and open problems

Question

What is true in this ω_1-Chang model $L(P_{\omega_2}\,\mathrm{Ord})$?
First of all many of the consequences of MM are Π^2_1-statements, if not even Π^1_2 over H_{ω_2}:
OCA (OGA?), All automorphisms of the Calkin algebra are inner, all Aronszjain trees are club isomorphic.......
One optimal solution would be obtained if the class of posets in the model which are stationary set preserving in V is definable in $L(P_{\omega_2}\,\mathrm{Ord})$.

"Foundational" questions...

Let $\phi(\Gamma, \lambda)$ assert that $\Omega_\lambda = \Gamma$ and:

> *For all* $\mathbb{P} \in \Gamma$ *and all Woodin cardinal* $\delta > |\mathbb{P}|$ *there is a regular embedding* $i : \mathbb{P} \to \mathbb{R}^\lambda_\delta$ *such that*
>
> $$\Vdash_{\mathbb{P}} (\mathbb{R}^\lambda_\delta)/i[\mathbb{P}] \in (\Gamma)^{V^{\mathbb{P}}}$$

Definition

A definable class of posets Γ is maximal for λ with respect to the theory T if T models the following:

1. $\Omega_\lambda \subseteq \Gamma$.
2. $\mathrm{Con}(T) \to \mathrm{Con}(T + \phi(\Gamma, \lambda))$.
3. $\mathbb{R}^\lambda_\delta \in \Gamma$ for all Woodin cardinals $\delta > \lambda$.
4. If $i : \mathbb{P} \to \mathbb{Q}$ is a locally complete embedding and $\mathbb{Q} \in \Gamma$, then $\mathbb{P} \in \Gamma$ as well.
5. If $\mathrm{Con}(T + \Gamma^* \setminus \Gamma \neq \emptyset)$ then $\Gamma^* = \Omega_\lambda$ is not consistent with T.

The following holds:

1. The class of all posets is maximal for $\mathsf{P}_{\aleph_1}$ relative to ZFC+ *there are class many Woodin cardinals*.
2. SSP is maximal for $\mathsf{P}_{\aleph_2}$ relative to ZFC+ *there are class many supercompact cardinals*.

Question

Relative to ZFC+ *large cardinal axioms*:

- can we simultaneously have a maximal class $\Gamma_{\aleph_3}$ for $\aleph_3$ and MM^{++}?.
- can we have a maximal class for $\aleph_3$ which contains all *CCC*-posets (and thus is incompatible with *MM*)?
- can we at all have a maximal class for $\aleph_3$?

Some requirements on $\Gamma_{\aleph_3}$ in case MM^{++} holds

Definition

$\mathbb{P}$ is (λ, κ)-distributive if $(P_\kappa \lambda)^V = (P_\kappa \lambda)^{V[G]}$ for any V-generic filter G for $\mathbb{P}$.

Definition

$\mathbb{P}$ is λ-SSP if

$$\Vdash_{\mathbb{P}} S \text{ is stationary}$$

for all stationary set $S \in H_\lambda$.

What can be in $\Gamma_{\aleph_3}$.

Lemma

Asume δ is a Woodin cardinal. $\mathbb{R}_\delta^{\aleph_3}$ is always $\aleph_3$-SSP and is $(\aleph_2, \aleph_2)$-distributive if $2^{\aleph_1} = \aleph_2$.

Magidor and I have two distinct counterexamples of posets that are $\aleph_3$-SSP and $(\aleph_2, \aleph_2)$-distributive that cannot be embedded into $\mathbb{R}_\delta^{\aleph_3}$ in case MM^{++} holds.

Prikry type forcings and $\aleph_2$-directed closed posets regularly embeds into $\mathbb{R}_\delta^{\aleph_3}$ for sufficiently large Woodin cardinals δ.

Question

Is the class Ω_λ closed under two-step iterations?

This is not clear even for $\Omega_{\aleph_2}$ if MM fails.....

THANKS FOR YOUR PATIENCE AND ATTENTION